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Major P. A. MACMAHON, D.Sc., Sc.D., LL.D., F.R.S., PRESIDENT, in the Chair.

Cyril Heath Stephenson, B.A., Brinkburne, Darley Green, Knowle, Warwickshire; and
William Wilson, M.B., C.M., Medical Missionary of the China Inland Mission, 43 Fellows Road, South Hampstead, N.W. 3,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

W. R. Cullington, Headmaster, Trinity House Navigation School, Hull (proposed by the Rev. William Smith);
The Rev. William Frederick Archdall Ellison, M.A., B.D., Rector of Fethard-with-Tintern (proposed by the Rev. T. H. E. C. Espin); and
Arthur Joseph Kelley, 16 York Road, Edgbaston, Birmingham (proposed by J. H. Reynolds).

Mr. David Baxandall, Mr. H. P. Hollis, and Mr. G. J. Newbegin were appointed Auditors of the Treasurer's accounts for the year 1917.

Fifty presents were announced as having been received since the last meeting, including, amongst others:—

The book of the opening of the Rice Institute; being an account in three volumes of an Academic Festival held in celebration of the formal opening of the Rice Institute . . . Houston, Texas, presented by the Institute.

Oxford University Observatory Miscellaneous Papers, edited by H. H. Turner, Vol. IV., 1910–14 (Nos. 175–228), presented by the Observatory.

On the Classification of Long-period Variable Stars.
By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. Some years ago I suggested a classification of long-period variables (*M.N.*, lxiv. p. 549, and lxvii. p. 334; *Mem. R.A.S.*, lv. p. xcvii), taking as the primary variant the phase of the first harmonic at maximum or, what comes to the same thing, the interval M − *m* (by which "maximum" follows "minimum," tabulated by Chandler in his catalogues) converted into the quantity

$$\alpha = \{2(\mathrm{M} - m) - \mathrm{P}\}/\mathrm{P},$$

so as to give comparable results for different periods (P). It was noted that stars with positive values of α did not occur near the Galactic Poles (*M.N.*, lxvii. p. 352), and a tentative physical explanation was given of this result: viz. that the variability being due to fluctuating "activity" similar to that of our own Sun, which first affects chiefly high latitudes and then approaches the equator, the stars with positive values of α were those with their poles of rotation turned towards us, those with large negative values of α being stars seen equatorially as we see our Sun, and that the axes were all roughly parallel to the Galaxy.

2. In his Presidential Address to the British Astronomical Association (*B.A.A. Journal*, 1916 October) the Rev. T. E. R. Phillips made a considerable advance in the direction of classification by analysing harmonically a much larger number of light curves. He has, however, adopted a new primary variant, for reasons which I fully approve. There are great difficulties in the accurate identification of "maximum," and even more so of "minimum" (see *M.N.*, lxvii. p. 337), and to get rid of these Phillips has taken his origin from the maximum of the first harmonic, which is thoroughly definite. [It was my own intention to adopt this origin in returning to the subject.] He thus reduces his light curves to the uniform expression

$$k_1 \cos(\theta - 180^\circ) + k_2 \cos(2\theta - \phi_2) + k_3 \cos(3\theta - \phi_3),$$

and his main results are obtained from the relation between ϕ_3 and ϕ_2. He finds that there are two groups of variables:—

Group I. Range of ϕ_2 from 233° to 126°: ϕ_3 nearly constant at about 200°; actual relation

$$\phi_3 = 202°{\cdot}1 - 0{\cdot}04\phi_2.$$

Group II. Range of ϕ_2 from 67° to 197° and of ϕ_3 from −60° to about 200°, sensibly varying with ϕ_2: actual relation

$$\phi_3 = 1{\cdot}67\phi_2 - 126°{\cdot}8.$$

No reason has appeared for questioning the reality of this division into two groups: quite the contrary. A suggestion will presently be made for transferring a few stars from Group II. to Group I., but Phillips has left this possibility well open.

3. His classification takes no account of the coefficients k_2 and k_3, or rather, they seem indifferent to it: except that k_2 is distinctly larger for Group II., which is a valuable support for the division into two groups. Now the character I had chiefly in mind depends on ϕ_2 and k_2 conjointly, and sometimes almost entirely on k_2. Let us rewrite the expression for magnitude as

$$\text{mag.} = -\cos\theta - l_2 \sin 2\theta + m_2 \cos 2\theta + l_3 \sin 3\theta + m_3 \cos 3\theta,$$

and consider first the term $m_2 \cos 2\theta$. Its maxima fall at 0°, 180°; and its minima at 90°, 270°. [There is an unfortunate tendency to confusion in dealing with stellar magnitude, which increases with faintness. We shall use inverted commas for brightness and omit them for magnitude. Thus when $\theta = 0$ the term $-\cos\theta$ is a "max." or a min.] Now 0° and 180° are the *positions* of the min. and max. of the first harmonic, and are therefore not disturbed by the term $m_2 \cos 2\theta$, which only emphasises one and reduces the other, without altering the interval between. But the term $l_2 \sin 2\theta$ has its greatest slope or change when $\theta = 0°$ or 180°, and displaces the max. one way and the min. the other, the amount of displacement depending on the coefficient l_2. We may take l_2 as a rough measure of the quantity α mentioned in § 1. A term $l_4 \sin 4\theta$ would have a similar effect, but is usually small, though it must not be ultimately overlooked. A term $l_3 \sin 3\theta$ would displace both max. and min. in the same direction without altering the interval. All these statements are, however, only approximate. For instance, when once the term $l_2 \sin 2\theta$ has altered the interval between max. and min. so that it is no longer 180°, then the term $m_2 \cos 2\theta$ produces an additional effect. The total effect of the second harmonic is considered at the end of this paper: but even then the results are liable to modification by higher harmonics, especially the 4th and 6th, etc. If we could estimate the position of the max. and min. accurately, this would be the best way of obtaining the quantity α: but there are great difficulties in doing this. To assess α

from the second harmonic as in Table IX. has the disadvantage of incompleteness, but the advantage of a numerical process, and will be adopted as the best we can do at present.

4. But before rearranging Phillips's groups according to α as is proposed, we can see in a general way what will be the effect by regarding α as roughly equivalent to l_2 (see Table I.). Since $l_2 = -(k_2 \sin \phi_2)/k_1$, it will be + when ϕ_2 lies between 180° and 360° and − when ϕ_2 lies between 0° and 180°. Phillips's groups may thus be assorted as follows :—

Group I. A.

$\phi_2 = 233°$ to $358°$: $\alpha +$; 10 stars.

The galactic latitudes of these stars are, in order $+13°$, $+38°$, $+19°$, $+14°$, $+8°$, $-21°$, $-8°$, $+28°$, $+68°$, $+58°$, the first 8 being small. For the last 2 the value of $l_2 = -(k_2 \sin \phi_2)/k_1$ is small, since ϕ_2 has values 345° and 358°. Consequently, the statement made (*M.N.*, lxvii. p. 352) that "stars with positive values of α do not occur near the Galactic Poles" is satisfactorily confirmed if we interpret "positive" to mean "decidedly positive."

Group I. B.

$\phi_2 = 7°$ to $126°$: l_2 or $\alpha -$; 18 stars.

The distribution in galactic latitude is

0° − 10°	− 20°	− 30°	− 40°	− 50°	− 60°	− 70°	− 80°	− 90°
1	2	4	3	2	2	3	1	

There are thus 7 with latitude less than 30° (including one of just 29°) and 11 greater (including one of 30°). Certainly no tendency to *low* latitudes.

Group II. C.

$\phi_2 = 67°$ to $160°$: l_2 or α definitely − ; 31 stars.

The distribution in galactic latitude is

0° − 10°	− 20°	− 30°	− 40°	− 50°	− 60°	− 70°	− 80°	− 90°
7	7	2	2	6	4	2	1	0

There are thus 16 below 30° and 15 above, no stars occurring very close to 30°.

When ϕ_2 exceeds 160°, the quantities l_2 and α are small, and may be uncertain. The stars have been divided as follows :—

Group II. D.

$\phi_2 = 162°$ to $169°$; 6 stars.

Galactic latitudes $+43°$, $-45°$, $+38°$, $-14°$, $+20°$, $+24°$; Mean $\pm 31°$.

Group II. E.

$\phi_2 = 176°$ to $197°$; 9 stars.

Galactic latitudes $+53°$, $-71°$, $+55°$, $-71°$, $+57°$, $-25°$, $-15°$, $+13°$, $+46°$; Mean $\pm 45°$.

5. Now, if the $\phi_2 = 90°$ end of Group II. corresponds to the $\phi_2 = 90°$ end of Group I., then we should look for *low* galactic latitudes at the other end of Group II. But we find on the contrary *high* galactic latitudes in the terminal set Group II. E.

The suggestion I wish to put forward is that II. E really belongs to Group I. There is a clear break of $7°$ in the values of ϕ_2 between II. D and II. E, whereas on both sides of this gap the values of ϕ_2 are in close sequence. The transference is attended by certain difficulties which will be duly noted; but if we make it tentatively the order now is:—

Class I.

Group I. B $\phi_2 = 7°$ to $126°$: $l_2 -$; 18 stars; gal. lat. $35°$.
Group II. E $\phi_2 = 176°$ to $197°$: l_2 o; 9 stars; gal. lat. $45°$.
Group I. A $\phi_2 = 233°$ to $358°$: $l_2 +$; 10 stars; gal. lat. $27°$.

But this is only a provisional grouping: the drop in galactic latitude will be clearer when the class is arranged according to value of l_2 and not according to ϕ_2 simply.

Class II.

This consists of Group II. shorn of II. E, so that ϕ_2 stops short at $167°$. When arranged in order of ϕ_2, as at present, it shows no drop in galactic latitude at the end. This is only shown when α is taken as primary variant: and we proceed to rearrange both classes in this way.

6. The chief difficulty is to get a good value of α. The quantity α_C is taken from Chandler's Revision of his 3rd Catalogue (*A.J.*, 553), using the formula

$$\alpha_C = \{2(M - m) - P\}/P,$$

where P is the period. To save decimals this and similar quantities are expressed in units of 0·01. But Chandler's determinations are now rather out of date, and estimates were made from the Harvard light curves given in *H.A.*, lvii. pp. 185–198, under the heading α_H. These data are convenient for the purpose. It may be fairly assumed that

(*a*) The "min." is either the largest magnitude recorded, or is nearer to it than to any other record.

(*b*) That when the adjacent records on opposite sides are equal, the "min." is *at* the largest number. When they are unequal, it is on the side of the greater.

These two assumptions, elementary as they are, are almost enough to get fair values for α_H, adopting the "max." as given by

Harvard. To avoid any possible bias the estimations were kindly made for me by Mr. F. A. Bellamy.

7. But, unfortunately, we have not either α_C or α_H for all the stars, and for completeness we must determine α from Phillips's data. The complete method is given at the end of this paper (see Table IX.). But to give an idea of the process let us assume here that the max. and min. are determined in position entirely from the terms

$$\text{mag.} = -\cos\theta - l_2 \sin 2\theta.$$

Then they are given by the equation

$$\sin\theta = 2l_2 \cos 2\theta \quad . \quad . \quad . \quad . \quad (L)$$

It is easy to tabulate from equation (L) the values of l_2 corresponding to different values of θ, and by a graphical method to deduce the relation between α and l_2 as in Table I.

TABLE I.

l_2.	α.	l_2.	α.	l_2.	α.	l_2.	α.
·02	·026	·10	·119	·18	·192	·26	·244
·04	·050	·12	·140	·20	·207	·28	·254
·06	·074	·14	·159	·22	·220	·30	·262
·08	·097	·16	·176	·24	·233	·32	·270

In Table II. the 9 stars imported from Group II. are in italics.

TABLE II.

Class I. arranged according to α.

Name.	Harv. No.	α_P.	α_C.	α_H.	Gal. Lat.	m_2.	ϕ_2.
T Cassiop.	001755	+24	+11	+ 7	− 8	+23	191
R Aurigæ	050953	+17	+ 2	+ 6	+ 8	+ 9	216
S Orionis	*052404*	+16	− 6	...	−21	+13	168
S Cygni	200357	+14	+ 1	+ 7	+13	− 8	206
T Cephei	210868	+10	+ 8	+ 5	+14	+ 4	205
R U Herculis	160625	+ 9	...	...	+46	−16	187
S Cephei	213678	+ 8	+10	+ 2	+19	+ 2	204
S Herculis	164715	+ 8	− 1	− 2	+38	+ 2	187
U Virginis	124606	+ 5	−15	− 1	+68	+12	197
S Hydræ	084803	+ 2	−22	− 3	+28	+ 4	189
R Ophiuchi	*170215*	+ 1	...	− 2	+13	− 4	195
W Androm.	021143*a*	+ 1	...	...	−15	−13	222
S Boötis	141954	0	− 2	− 2	+58	+ 7	258
R Tauri	042209	0	−13	− 7	−25	−21	107
V Virginis	*132202*	− 1	...	− 7	+57	− 8	182
S Ceti	001909	− 1	−10	− 8	−71	−13	195
R Virginis	123307	− 2	− 6	− 2	+69	+ 8	216
R Camelop.	142584	− 2	+ 5	− 1	+33	+11	213
R Boötis	143227	− 3	−10	− 2	+66	− 2	202

TABLE II.—*continued.*

Name.	Harv. No.	a_P.	a_C.	a_H.	Gal. Lat.	m_2.	ϕ_2.
S Urs. Maj.	123961	− 5	− 5	− 3	+55	+17	263
S Libræ	*151520*	− 5	− 3	−15	+30	+ 6	186
R Arietis	021024	− 5	− 2	− 3	−34	+14	197
R Lyncis	065355	− 5	− 2	− 8	+24	+ 4	211
S Serpentis	151714	− 6	...	− 5	+53	−28	125
R Can. Ven.	134440	− 6	...	− 2	+72	+15	205
T Herculis	180531	− 6	− 4	− 7	+22	+ 4	160
R Vulpec.	205923	− 7	−10	− 7	−16	+ 6	153
T Virginis	120905	− 7	−10	−20	+55	−31	159
S Can. Min.	072708	− 9	− 1	− 7	+13	+10	207
V Androm.	004435	−12	...	...	−27	+ 1	169
R Aquarii	*233815*	−12	...	−12	−71	−44	160
U Cygni	201647	−13	− 1	−11	+ 6	+15	188
R Leonis	094211	−13	− 8	− 9	+42	− 5	154
U Ceti	*022813*	−14	+ 4	−13	−61	+ 9	241
R Ceti	*022000*	−16	−16	−20	−55	+ 3	226
T Draconis	175458	−16	−14	...	+29	+ 1	18_3
R Serpentis	154615	−16	−15	...	+46	+ 2	248

8. Collecting the material into groups as indicated, we get the means of Table III.

TABLE III.

Groups for Class I.

No. of Stars.	a_P.	a_C.	a_H.	Gal. Lat.	m_2.	ϕ_2.
8	+13	+4	+ 4	±21°	+4	196°
11	0	−9	− 3	±46	−2	198
9	− 6	−5	− 8	±40	−1	184
9	−15	−7	−12	±39	−5	197

The galactic latitudes for the first group are thus distinctly small. The galactic longitudes were also examined, but showed no special features, and are omitted from the table.

9. It seemed desirable to tabulate the coefficient $m_2 = (k_2 \cos \phi_2)/k$, and its value is given in units of 0·01; but when the stars are rearranged in order of m_2, the galactic latitudes become indifferent. Seven groups gave the following means:—

m_2.	+17.	+11.	+7.	+4.	+2.	−5.	−24.
Gal. Lat.	±35°	±38°	±37°	±29°	±32°	±38°	±48°
No. Stars	5	5	5	5	5	5	7

10. Accidental error may of course be accountable for the small latitude of the first group in Table III. This objection can only be removed by further observation. A list of 20 stars worthy of examination was given in *M.N.*, lxvii. p. 351, of which only 6 appear in Phillips's lists, as shown in Table IV.

TABLE IV.

Name.	α_P.	α_C.	Gal. Lat.	Name.	α_P.	α_C.	Gal. Lat.
			°				°
T Cassiop.	+24	+12	− 8	S Cephei	+ 8	+10	+19
R Aurigæ	+17	+ 3	+ 8	R Camelop.	− 2	+ 5	+33
T Cephei	+10	+ 8	+14	[R Sagittarii	− 3	+ 2	−15]

It is noteworthy that the largest galactic latitude + 33° has gone down the list, with improved α. The last star is in Class II., and appears in Table VIII. There remain the following stars, worthy of examination because of their provisional positive values of α. They have been rearranged in order of galactic latitude, and we should expect those at the top of the list to retain or improve their $+\alpha$, those at the bottom to lose it, in part at least, under further observation.

TABLE V.

Stars specially worthy of observation.

Name.	R.A.	Gal. Lat.	α_C.	Name.	R.A.	Gal. Lat.	α_C.
	h m	°			h m	°	
V Cygni	20 38	+ 3	+ 5	W Scorpii	16 6	+22	+18
S Carinæ	10 6	− 4	+16	V Ophiuchi	16 21	+24	+17
W Cassiop.	0 49	− 5	+ 3	RS Libræ	15 18	+27	+19
X Ophiuchi	18 34	+ 6	+25	X Capric.	21 3	−40	+ 7
W Cygni	21 32	− 6	+ 7	T Capric.	21 17	−42	+11
V Tauri	4 46	−16	+ 8	S Arietis	1 59	−46	+13
V Cephei	23 52	+21	+22	U Boötis	14 50	+60	+ 7

11. In the eighth column of Table II. are shown Phillips's values of ϕ_3. His important discovery of the constancy of ϕ_3 is of course not disturbed. But it is important to note that the 9 imported stars agree better, both as regards phase ϕ_3 and coefficient k_2, with Group II. rather than with Group I. The mean value of ϕ_3 is only 170°, though it would rise to 178° if we omit R Tauri, which is discordant with either Group I. or Group II. The mean value of k_2, the coefficient of the 2nd harmonic, is 0·55, sensibly higher than the value 0·32 found by Phillips for the mean of Group I. These objections may not be fatal, but they are serious enough; and against them are only to be set, first, the separating gap of 7° in ϕ_2 already mentioned, and, second, the fact that it is just possible to re-state the proposition about low galactic latitudes, formerly made without any suspicion of the division into two groups, under the new and unexpected condition that it should be applicable to each group separately. Some attempts

were made to remove or lessen these objections by finding possible reasons for the discrepancies: for instance, the stars were re-arranged in order of m_2, and there appeared some grounds for suspecting a drop in ϕ_3 depending on m_2, as also a continuous change in k_2, also depending on m_2. On these lines the objections could certainly be reduced; but it seems preferable to leave them as they stand, to be dealt with rather by the evidence of future observations than by elaborating hypotheses at the present juncture.

12. One more point deserves examination before passing on to Class II. If we adopt the physical interpretation that these variables are rotating suns with axes parallel to the Galaxy, then two conditions should be fulfilled:—

(*a*) Stars of which we see the poles should occur only in low galactic latitudes, though in these latitudes there may also be stars of which we see the equators.

(*b*) In high galactic latitudes we should only see equatorial stars, though these stars may also occur in other latitudes.

The first condition has been examined. For the second we pick out from Table II. the highest galactic latitudes as in Table VI.

TABLE VI.

Gal. Lat.	α_P.	α_C.	α_H.	Gal. Lat.	α_P	α_C.	α_H.
+72°	− 6	...	− 2	−61°	−14	+ 4	−13
−71	−12	...	−12	+58	0	− 2	− 2
−71	− 1	−10	− 8	+57	− 1	...	− 7
+69	− 2	− 6	− 2	+55	− 7	−10	−20
+68	+ 5	−15	− 1	+55	− 5	− 5	− 3
+66	− 3	−10	− 2	−55	−16	−16	−20

The most discordant star in both Tables II. and VI. is U Virginis 124606, galactic latitude +68°. Phillips records only one curve analysed, which is doubtless the Harvard Curve, so that α_P and α_H naturally tend to agree; but it will be seen that α_C is very different, and perhaps this star might receive further attention.

13. We now pass to the consideration of Class II. Table VIII. is arranged on the same lines as Table II. in order of α_P deduced from k_2/k_1 and ϕ_2 by means of Table IX. It will be seen that the first 5 stars have small galactic latitudes. We may group the material as in Table VII.

TABLE VII.

Class II. in groups according to α_P.

No. of Stars.	α_P.	α_C.	α_H.	Gal. Lat.	m_2.	ϕ_2.
5	− 4	−12	− 3	±14°	− 7	119°
9	− 9	−14	− 9	±30	− 9	120
7	−17	−22	−20	±31	− 6	80
8	−20	−22	−21	±33	−14	100
8	−24	−28	−26	±39	−15	77

14. The values of ϕ_3 diminish with a_P, but not very regularly. Moreover, in forming the means in the last column three quite discordant stars have been omitted, viz. R Persei, T Ursæ Majoris, S Lyræ. A formula was constructed for deducing ϕ_2 from a_P, but the residuals were not so good as those given by Phillips's formula, and it seems clear that ϕ_3 is more closely related with ϕ_2 than with a. The stars were rearranged in order of m_2, but nothing particular emerged from the rearrangement. The case for the redistribution by transferring 9 stars is thus not strong, but seems worth stating as a guide to further test observations.

TABLE VIII.

Class II. arranged according to a_P.

Name.	Harv. No.	a_P.	a_C.	a_H.	Gal. Lat.	m_2.	ϕ_3. °
Ṙ Sagittarii	*191019*	− 3	+ 3	− 6	−15	− 3	94
S Scorpii	*161122b*	− 3	...	+ 1	+20	−11	140
R Cancri	081112	− 5	−31	− 3	+24	−16	154
SX Cygni	201130	− 5	...	...	− 4	− 7	89
R Persei	032335	− 5	− 9	...	−17	+ 1	(336)
R Draconis	163266	− 7	−12	− 7	+38	− 4	60
S Cassiop.	011272	− 7	− 8	−14	+ 9	− 6	107
R Hydræ	*132422*	− 8	−10	−12	+38	−17	181
R Aquilæ	190108	− 8	−22	...	0	− 6	89
S Virginis	*132706*	− 9	−17	− 5	+53	−14	161
W Herculis	163137	− 9	− 9	− 9	+41	−10	125
R Delphini	201008	−10	− 9	− 5	−14	−15	171
T Androm.	001726	−10	...	− 7	−36	− 3	80
R Herculis	160118	−11	...	−11	+44	− 9	102
R Leo. Min.	093934	−13	−11	−18	+49	−12	106
T Urs. Maj.	123160	−14	−16	−16	+57	− 1	(356)
U Orionis	054920	−17	−21	−17	− 2	−11	106
R Urs. Maj.	103769	−17	−27	−21	+44	−11	75
R Gemin.	070122*a*	−18	−34	−26	+14	− 3	35
S Pegasi	231508	−19	−13	−19	−48	− 5	93
Z Cassiop.	233956	−19	...	...	− 5	− 2	65
R Scorpii	*161122a*	−20	...	−18	−19	−17	137
R Corvi	*121418*	−20	...	−10	+43	−25	147
R Androm.	001838	−20	−42	−23	−24	− 2	66
U Herculis	162119	−20	−17	−12	+41	−22	111
R Cygni	193449	−20	−27	−24	+13	− 5	93
R Pegasi	230110	−21	− 9	−26	−45	−27	133
R Cassiop.	235350	−21	−16	−22	−10	−10	89
S Aquarii	*225120*	−21	...	−29	−65	− 5	21

TABLE VIII.—*continued.*

Name.	Harv. No.	a_P.	a_C.	a_H.	Gal. Lat.	m_2.	ϕ_2.
							°
R Com. Ber.	115919	−22	−34	−36	+78	−12	33
o Ceti	*021403*	−22	−25	−28	+64	−11	59
S Coronæ	151731	−22	−34	−18	+57	− 6	69
S Lyræ	190925	−23	+ 6	...	+ 6	+10	(299)
V Delphini	204318	−23	...	...	−16	− 7	63
R Piscium	012502	−24	−17	−15	−58	−27	113
Y Cassiop.	235855	−25	...	...	− 7	−27	95
S Tauri	042309	−32	−63	−33	−25	−39	105

15. We now pass to the consideration of two different points which may influence ideas of classification.

The periodicity of sun-spots has recently been examined in a novel manner by Larmor and Yamaga (*Proc. R.S.*, A xciii. p. 493), who show that if the observations from 1776 to 1798 be omitted or neglected, the remainder can be fitted with considerable closeness to a uniform period, with a characteristic type of curve. Whatever we may think of the suggestion, to neglect the observations of a quarter of a century in order to establish the uniformity of a period, we need not hesitate to accept the valuable determination of the type of curve, which would be but little affected if the rejected observations were included in any reasonable manner. The authors arrive at a result which is not only elegant but may, as they remark, be of physical significance. They give the alternative expressions

$$44{\cdot}5 + 35{\cdot}4 \sin kt + 6{\cdot}6 \sin 2kt$$

or

$$45 + 37{\cdot}8 \sin k't + 6{\cdot}5 \sin 2k't + 1{\cdot}4 \sin 3k't,$$

the harmonic terms being in either case all in the same phase, or, to put it in another way, there are no cosine terms required. The result is admitted to be only approximate: actual calculation gives cosine terms with coefficients less than unity, and it is only by excluding these as probably accidental that the above clear-cut expressions are obtained. But there is nothing unreasonable in this procedure.

16. Some years ago I examined the nature of the sun-spot variation in connection with the suggested classification of long-period variables (*M.N.*, lxiv. p. 549 and lxvii. p. 334), coming to the conclusion that there was an essential difference in the possibility of connection when the curve was turned upside down so that maximum became minimum. There are two possible views of sun-spot variation considered as an index of stellar variability, viz.:—

Hypothesis A. Sun brighter when spotted, owing to the greater activity manifested in accompanying faculæ, etc. etc.

On this hypothesis the following expression was obtained for the variation (multiplied by a factor 6):—

$$-1{\cdot}49\sin\theta - 2{\cdot}32\cos\theta + 0{\cdot}11\sin 2\theta - 0{\cdot}70\cos 2\theta + 0{\cdot}19\sin 3\theta - 0{\cdot}19\cos 3\theta,$$

which agreed fairly well with a formula derived from variable stars, viz.

$$(-1{\cdot}49)\sin\theta - 2{\cdot}29\cos\theta - 0{\cdot}16\sin 2\theta - 0{\cdot}64\cos 2\theta + 0{\cdot}05\sin 3\theta - 0{\cdot}15\cos 3\theta,$$

where the first coefficient was adopted and then used to determine all the others.

Hypothesis B. Sun fainter when spotted, owing to darkness of spots.

On this hypothesis the observed expression was

$$+1{\cdot}51\sin\theta - 2{\cdot}35\cos\theta - 0{\cdot}46\sin 2\theta - 0{\cdot}54\cos 2\theta - 0{\cdot}23\sin 3\theta + 0{\cdot}14\cos 3\theta,$$

and as before, using the first coefficient $+1{\cdot}51$ to obtain the others from the variable star formula, we get

$$(+1{\cdot}51)\sin\theta - 1{\cdot}67\cos\theta + 0{\cdot}17\sin 2\theta - 0{\cdot}63\cos 2\theta + 0{\cdot}43\sin 3\theta - 0{\cdot}08\cos 3\theta,$$

whence it was concluded that the Sun would fit in with Hypothesis A but not with Hypothesis B.

17. Now Larmor and Yamaga say that to turn the sun-spot curve upside down makes no difference, because it is "antisymmetrical." This might seem inconsistent with the result just quoted, but is not really so. For a glance at the figures shows that the difference between A and B arises chiefly from the variable star formula, and not from the sun-spot variation. If we alter the zero from which θ is measured so as to make the coefficient of $\cos\theta$ zero, and multiply throughout by a suitable factor, the above expressions for the sun-spot curve become

Hyp. A. $36\sin\theta - 8{\cdot}8\sin 2\theta - 3{\cdot}4\sin 3\theta + 2{\cdot}3\cos 2\theta - 2{\cdot}1\cos 3\theta.$
Hyp. B. $36\sin\theta + 8{\cdot}8\sin 2\theta + 3{\cdot}4\sin 3\theta + 2{\cdot}3\cos 2\theta - 2{\cdot}1\cos 3\theta.$

All the coefficients after the first, which was arbitrarily chosen to resemble that of L. and Y., are larger than found by them, but are of the same order of magnitude. The result of L. and Y. is to be preferred, since it is the outcome of a more careful discussion of greater material, my own figures being deduced from the years 1849–1892 only.

18. For comparison with Phillips's formula we throw that of L. and Y. into the form

$$k_1\cos(\theta - 180^\circ) + k_2\cos(2\theta - 90^\circ) + k_3\cos(3\theta - 0),$$

or, rather, this is the form of their expression inverted to correspond with Hypothesis A (Sun brighter when spotted). Thus

$$\phi_2 = 90^\circ \qquad \phi_3 = 0^\circ.$$

The value of ϕ_3 makes it clear that the Sun cannot belong to Group I., and the value of ϕ_3 when $\phi_2 = 90^\circ$ is by Phillips's formula for Group II.

$$\phi_3 = 150^\circ - 127^\circ = 23^\circ,$$

not very different from zero. Inspection of Phillips's Table III. for Group II. shows that the Sun would come very near the beginning of the group with $\phi_2 = 90^\circ$. (Phillips used my value $\phi_2 = 106^\circ$, which puts the Sun further down.) The question suggests itself: Are the values $\phi_2 = 90^\circ$ and $\phi_3 = 0^\circ$ physically significant of the commencement of the group? The clear-cut figures are very tempting, especially since the other end of Group II. seems to lie very near

$$\phi_2 = 180^\circ \qquad \phi_3 = 180^\circ.$$

The relation between ϕ_2 and ϕ_3 would then be

$$\phi_3 = 2\phi_2 - 180^\circ.$$

Calculating ϕ_3 from ϕ_2 with this formula for Phillips's Group II. we naturally find an increase in the mean residual, which rises to $\pm 24^\circ$ instead of $\pm 22^\circ$, but the increase is not very great.

19. Leaving the Sun for a moment, let us return to the effects of the full second and third harmonics on the shapes of the light curves. In § 7 a rough assumption was made that the interval between max. and min. was determined entirely by the term $l_2 \sin 2\theta$, so that we had to solve the equation

$$\sin\theta = 2l_2 \cos 2\theta \quad . \quad . \quad . \quad . \quad \text{(L)}$$

But a more accurate, though still incomplete, assumption would take in the whole of the second harmonic, writing

$$\text{mag.} = -\cos\theta - n_2 \cos(2\theta - \phi_2)$$

and leading to the equation

$$\sin\theta = 2n_2 \sin(2\theta - \phi_2) \quad . \quad . \quad . \quad \text{(N)}$$

where $n_2 = k_2/k_1$. This is no more difficult to solve than equation (L), but more tiresome, involving tabulation in two co-ordinates, n_2 and ϕ_2. The numerical solution is as in Table IX.

TABLE IX.

a in units of 0·01 ; *determined from* n_2 *and* ϕ_2 *only.*

Positive for $\phi_2 > 0° < 180°$ as at top of table ; negative at foot.

ϕ_2	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
n_2	180	170	160	150	140	130	120	110	100	90
·05	0	1	2	3	4	4	5	6	6	7
·10	0	2	4	7	8	9	10	11	12	12
·15	0	4	8	12	13	14	15	15	16	17
·20	0	8	13	17	18	19	19	19	20	21
·25	0	16	19	21	22	22	23	23	24	24
·30	0	21	23	24	25	25	26	26	26	27
·35	0	25	26	27	28	28	28	28	28	28
·40	0	28	29	30	30	30	30	30	30	30
·45	0	31	31	32	32	32	32	32	32	32
·50	33	33	33	33	33	33	33	33	33	33

n_2 / ϕ_2	180°	190°	200°	210°	220°	230°	240°	250°	260°	270°
	360	350	340	330	320	310	300	290	280	270

20. The equation (N) has several roots, and the proper choice is most readily made by considering a graphical solution. We have to find the intersections of the two curves $y = \sin\theta$ and $y = 2n_2 \sin(2\theta - \phi_2)$.

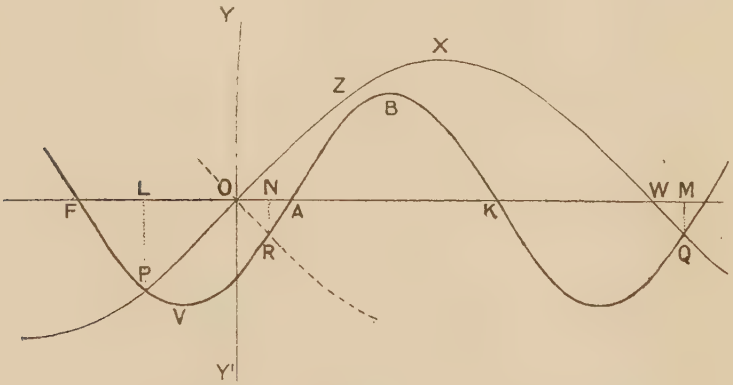


FIG. 1.

Let the first curve $y = \sin\theta$ be OXW, cutting the axis of θ at the origin O where $\theta = 0°$. The second curve starts from a point A where $\phi_2 = 2OA$, and oscillates twice as often. (See Fig. 1.)

If the second harmonic were non-existent, or if $n_2 = 0$, the second curve would be simply the horizontal line LOW cutting

the first curve at O and W, the point O corresponding to the "max." and W to the "min." The existence of the second harmonic transfers the intersection O to P, thus putting the "max." back by the amount OL, and it transfers the intersection W to Q, thus putting the "min." forward by WM. Thus $m - M = 180^\circ + OL + WM$.

21. But it will be seen that if the second curve had been drawn with the point A a little nearer to O, there would have been two other intersections near the points B and Z, corresponding to secondary "min." and "max." These intersections would also occur if the second curve, without disturbing the point A, were a little higher—that is, if the coefficient $2n_2$ were larger. On the other hand, it can be seen that if the second curve were flatter there could be no secondary intersections. For this to happen it must be so flat that the slope at the point A is less than the slope of the first curve at O; then on sliding A up towards O, the hump at B would not reach the curve OZX until after A had reached O. In other words, there are no secondary intersections if

$$4n_2 < 1,$$

and there are always secondary intersections if

$$2n_2 > 1.$$

22. For the variable stars tabulated by Phillips the largest values of n_2 are those for S Tauri and R Aquarii, which are $1{\cdot}15/2{\cdot}55 = {\cdot}45$ and $1{\cdot}09/2{\cdot}48 = 0{\cdot}44$, neither of which quite reaches the second limit. But they are well above the first limit, and several other stars have values of n_2 above that limit. The general question of secondary maxima thus merits attention, and we shall presently see its importance. But confining attention to the primary intersections for the moment, remark that if from O we draw the arc OR precisely similar to WQ, then ON = WM, and $m - M = 180^\circ + LN$. If, now, we imagine the portion AVF to slide along the axis AF while the legs OP, OR remain fixed, we can see how the interval LN varies. It will not change much until A or F are near the point O, and this fact is clearly represented in the lower part of Table IX., where n_2 is large. When A approaches O, the part ON shrinks, but the part OL compensates the shrinkage by lengthening, provided the slope of OP is less than that of AR, as drawn in the figure. For smaller values of n_2 the flatness of AR stops this lengthening, and both OL and ON shrink, as shown in the higher part of Table IX.

23. The case $2n_2 = 1$ is not represented among variable stars, but is worth attention. It is represented in the lowest line of Table IX., the value of α being shown constant right up to $\phi_2 = 0^\circ$ or 180°, though there is a *per saltum* change at those values of ϕ_2 from $\alpha = +{\cdot}33$ to $\alpha = -{\cdot}33$. For this particular case we can explicitly solve the equation (N), which becomes

$$\sin\theta = \sin(2\theta - \phi_2).$$

It will be seen that the four solutions of this are as follows:—

Primary "max." $2\theta - \phi_2 = -\pi - \theta$ or $\theta = (\phi_2 - \pi)/3$.
Secondary "min." $2\theta - \phi_2 = \theta$ or $\theta = \phi_2$.
Secondary "max." $2\theta - \phi_2 = \pi - \theta$ or $\theta = (\phi_2 + \pi)/3$.
Primary "min." $2\theta - \phi_2 = 3\pi - \theta$ or $\theta = \pi + \phi_2/3$.

Thus $m - M = \pi + \phi_2/3 - (\phi_2 - \pi)/3 = 4\pi/3$, which is independent of ϕ_2.

24. But when ϕ_2 changes through 0 or π, the long interval changes from $m - M$ to $M - m$. How is this sudden change effected? We can easily see what happens by tracing a few cases; say, those for $\phi = \pi/2$, $3\pi/4$, and π, as in fig. 2.

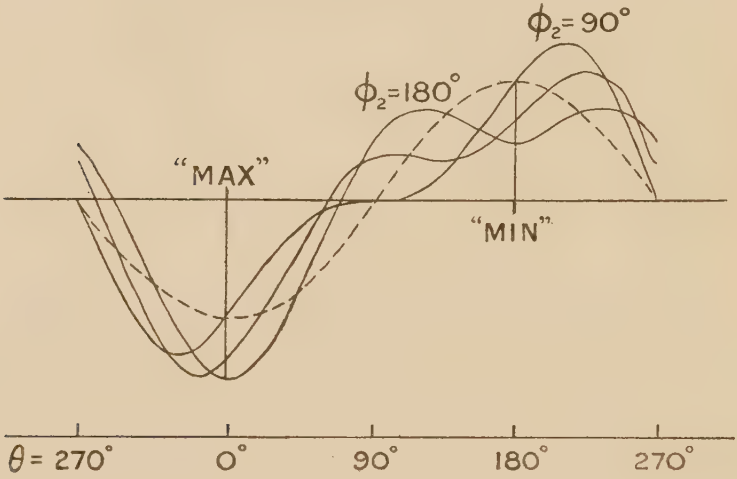


FIG. 2.

25. First look at the curve $\phi_2 = 90°$. The "max." and "min." are similar, each being displaced 30° from their original positions. At the mid-point is a point of inflexion, the coalescence of the secondary max. and min.

For the intermediate curve $\phi_2 = 135°$ the displacement of the max. has increased to 45°, that of the min. fallen to 15°; but the sum remains 60° as before. The secondary min. and max. are now well separated, but distinctly inferior to the primary.

But in the curve $\phi_2 = 180°$ the secondary max. has become equal to the first. With a slightly greater value of ϕ_2 the secondary will become the primary. The curve is thus in the act of changing maxima. If we stick to the first, displaced 60° to the left, we get $m - M = 240°$ as before; but when we change to the secondary, the displacement is 60° to the right, and $m - M = 120°$.

26. And we can take another view of the position of the max. We can say that it lies between the two humps in its

original position, but that there is a dip at maximum. For small values of n_2 the max. has no actual dip but becomes long and flat. But there will be an actual dip when n_2 exceeds 0·25, while ϕ_2 is at the same time near 0° or 180°. It is interesting, therefore, to examine the light curves of the following three stars:—

Harv. No.	Name.	n_2.	ϕ_2. °
151714	S Serpentis	0·28	176
233815	R Aquarii	0·44	176
120905	T Virginis	0·32	177

and especially that of R Aquarii, which ought to show a notable dip. On looking at the Harvard figures in *H.A.*, lvii., no such phenomenon appears. There is a *protracted* max. (="min."), but no secondary min. or max. Have they been smoothed out, or does the star really show no such phenomena? The case is worth examining in some detail. In the second column of Table X. are given the Harvard readings for twenty-four equi-

TABLE X.

Comparison of Harvard Mags. of R Aquarii with Calculation.

(The unit is 0·01 mag.)

	Harv.	(1).	(2).	S_2.	H − S_2.	(3).	S_3= S_2+(3).	H − S_3.
0	663	−165	−62	773	110	−20	753	90
1	700	−169	−69	762	62	−22	740	40
2	753	−163	−57	780	27	−12	768	15
3	808	−144	−30	826	18	+ 6	832	24
4	866	−117	+ 5	888	22	+20	908	42
5	923	− 80	+39	959	36	+22	981	58
6	978	− 40	+62	1022	44	+12	1034	56
7	1024	+ 2	+69	1071	47	− 6	1065	41
8	1050	+ 47	+57	1104	54	−20	1084	34
9	1060	+ 88	+30	1118	58	−22	1096	36
10	1065	+123	− 5	1118	53	−12	1106	41
11	1068	+149	−39	1110	42	+ 6	1116	48
12	1071	+165	−62	1103	32	+20	1123	52
13	1073	+169	−69	1100	27	+22	1122	49
14	1073	+163	−57	1106	33	+12	1118	45
15	1071	+144	−30	1114	43	− 6	1108	37
16	1065	+117	+ 5	1122	57	−20	1102	37
17	1055	+ 80	+39	1119	64	−22	1097	42
18	1040	+ 40	+62	1102	62	−12	1090	50
19	1018	− 2	+69	1067	49	+ 6	1073	55
20	987	− 47	+57	1010	23	+20	1030	43
21	940	− 88	+30	942	2	+22	964	24
22	865	−123	− 5	872	7	+12	884	19
23	764	−149	−39	812	48	− 6	806	42

distant values of θ, the next columns, headed (1) and (2), show the calculated first and second harmonics, and the fifth column shows their sum, adding 1000 so as to make all the readings comparable with Harvard. In this sum the secondary min. and max. are clearly visible, the former being between terms 9 and 10, and the latter at term 13. The discrepancies with observation are shown in column 6, headed $H - S_2$. But now add the calculated third harmonic as in column 7, forming the new sums as in column 8. The secondary min. and max. disappear, and there is now a single protracted max. just after term 12. The last column shows the final difference from Harvard.

27. These facts give rise to the following considerations:—

First suppose that the variation of the star were due to two independent causes, as in the case of the "equation of time." Then the superposition of the second on the first would produce a secondary "min." and max.," as we have seen. It seems improbable that a third cause should now be superimposed on the former two in such a way as just to destroy these phenomena, and yet the observations would support this view.

28. But, secondly, suppose that the analysis into harmonics is merely artificial; suppose, in fact, that the primary character of the variation is a smooth rise to max. and a smooth fall to min., the latter taking sensibly longer than the former. When we try to represent this by harmonics, the first two terms introduce inevitably a secondary min. and max. which are not shown by the observations. The calculated curve cannot fit the observations so long as we have only two harmonics; we must add a third, and one important part of its functions is to take out the spurious "min." and "max." We have seen how easily it does this. But if its function is thus to correct errors introduced by the second harmonic, we begin to see how it may be dictated by the second harmonic—that is, we may have a clue to the connection between ϕ_2 and ϕ_3, which Phillips has demonstrated.

29. Let us consider Phillips's Group II. of variable stars from this point of view. We start with the stars (including the Sun) for which ϕ_2 is approximately 90°. The curves are antisymmetrical, but also skew, that is to say, the intervals separating max. and min. are unequal. The curve is generally similar to that labelled $\phi_2 = 90°$ in fig. 2; but it need not have the point of inflexion, which is simply due to the fact that we have in that figure used only the first two harmonics. With these only, the best approximation we can make to a skew antisymmetrical curve will inevitably have this defect in the middle. It may not be so large as in fig. 2, where we have used a specially large second harmonic; but there will be traces of such a defect, and if we calculate a third harmonic, one of its duties must be to remedy this defect if possible. The curve being antisymmetrical will naturally remain so, and the third harmonic will cross at the mid-point, where $\theta = 90°$, from below to above. Hence the equation takes the form

$$\text{mag.} = \cos(\theta - 180°) + n_2 \cos(2\theta - 90°) + n_3 \cos(3\theta - 0°).$$

30. Now consider the case $\phi_2 = 180°$, which has a double max. If the third harmonic remedies this defect, it must naturally counteract the dip at $\theta = 180°$, having its greatest value then. Hence the curve takes the form

$$\text{mag.} = \cos(\theta - 180°) + n_2 \cos(2\theta - 180°) + n_3 \cos(3\theta - 180°).$$

We thus find an explanation for the terminal values of Group II. suggested in § 18, viz.

$$\phi_2 = 90° \qquad \phi_3 = 0°$$
$$\phi_2 = 180° \qquad \phi_3 = 180°$$

and it seems reasonable that intermediate values of ϕ_2 should require intermediate values of ϕ_3, though it is not so easy to specify the exact value of ϕ_3 in the intermediate cases.

31. If it be the chief duty of the third harmonic to correct the defects of the second, then the coefficients ought also to be related. The mean values for Group II. are 0·52 and 0·21, ratio 0·40. Let us now arrange the values in order of k_2, and multiply it by 0·40 as the theoretical (C) value of k_3, putting alongside it the observed value (O), as below (unit 0·01). It will be seen that there is a fair

C.	O.	O−C.	C.	O.	O−C.	C.	O.	O−C.	C.	O.	O−C.
50	45	− 5	26	10	−16	22	30	+ 8	14	23	+ 9
36	30	− 6	26	25	− 1	20	28	+ 8	14	13	− 1
32	36	+ 4	26	16	−10	20	24	+ 4	14	4	−10
30	61	+31	24	18	− 6	18	12	− 6	11	23	+12
28	13	−15	23	14	− 9	17	17	0	11	14	+ 3
27	28	+ 1	23	34	+11	17	27	+10	10	36	+26
27	24	− 3	22	11	−11	17	14	− 3	9	11	+ 2
27	27	0	22	20	− 2	16	13	− 3	8	4	− 4
26	19	− 7	22	17	− 5	16	13	− 3	5	13	+ 8
									4	19	+15
Mean	31	31	0	24	19	− 5	18	20	+ 2	10	16	+ 6

case for relation between k_3 and k_2. The 9 stars at the end have been kept separate; for these the figures are

C =	44	35	32	25	18	16	15	9	4
O =	42	24	26	18	21	11	8	7	18
O−C =	−2	−11	−6	−7	+3	−5	−7	−2	+14

There are here 7 negative residuals to 2 positive. The mean ratio for these stars is in fact only 0·36 instead of 0·40, and the difference is emphasised a little if we compare this group with the 10 stars immediately preceding, for which ϕ_2 lies between 150° and 170°, the mean ratio being 0·25 for these. Does this difference support the allocation of the last 9 stars to Group I.? There is

something to be said for excluding 2 stars with exceptional residuals +31 and +26 above, viz. SX Cygni and R Pegasi, but the material scarcely warrants any but rough inferences at present.

32. But now, turning to Group I., we find a different state of affairs, as we see specially when $\phi_2 = 90°$. Instead of ϕ_3 being near zero it is near 180°; the second harmonic is thus of the opposite sign, and instead of removing any tendency to inflexion must emphasise it. In other words, the inflexion is already there, in the light curve, and is even greater than the second harmonic can supply, incidentally to its primary duty of representing the inequality of intervals M to *m* and *m* to M. Hence help is required from the third harmonic.

33. To examine what happens more particularly the following procedure was adopted. The means of 5 light curves, viz. those for T Herculis, V Ceti, R Ceti, R Leonis, R Boötis, were formed from the Harvard figures in *H.A.*, lvii. Then the first and second harmonics were formed and subtracted from this curve. The residuals gave a third harmonic, which was then multiplied by 5 and added to the original light curve so as to exaggerate its features. The resulting curve was as in fig. 3.

FIG. 3.

The characteristic feature is thus a pause on the rise just before max. and a pause on the fall just before min., greatly exaggerated in the above figure. Since both these pauses are required, we can see why the phase ϕ_3 should be undisturbed by changes in ϕ_2, for in general the max. and min. are displaced in opposite directions. The first "pause" is thus, so to speak, pulled one direction, the second "pause" in the opposite, and as a compromise the phase ϕ_3 remains undisturbed.

Summary.

34. We may now sum up the results of this discussion. Two alternative methods of classifying long-period variable stars have been proposed.

(*a*) Some years ago I took the interval M − *m* tabulated by Chandler as a basis. It depends chiefly on the *second* harmonic, but may be designated briefly by the quantity $\alpha = \{2(M - m) - P\}/P$.

(ϕ_3) Phillips has recently pointed out that the phase ϕ_3 of the *third* harmonic has notable features. It divides the stars into two groups, in one of which ϕ_3 is nearly constant, in the other it is proportional to ϕ_2. This division into groups was not suspected with method (α), but it seems to be of prime importance.

The two methods might conceivably have been different views of the same classification, but this does not appear to be the case. There seems to be no reason as yet, however, why both should not be useful.

35. In connection with method (α), the point formerly made was that stars with definitely positive values of α were all in low galactic latitudes, and this still remains compatible with the evidence. Further test of it is therefore the more desirable, and a list of stars of which good light curves would be specially welcome is given in Table V. At present the values of α are rough, and more precision is needed.

It may be well to show here the general effect on the above hypothesis of the increased precision due to Phillips's recent work. In *M.N.*, lxvii. p. 335, is given the material then available, from Rousdon or Parkhurst observations. The order is determined not by α itself, but by a quantity related to it, and the galactic latitudes of the first 11 stars are

Gal. Lat.	− 8°	+14°	+38°	+58°	+ 8°	+55°	+13°	+24°	+33°	+19°	+38°
α (Phillips).	+24	+10	+ 8	0	+17	− 5	+14	− 5	− 2	+ 8	− 7

[The stars can easily be identified, if required, in Tables II. and VIII. from the latitudes and the values of α_P subjoined.] Now, rearranging them according to Phillips's improved determinations we get

Gal. Lat.	− 8°	+ 8°	+13°	+14°	+19°	+38°	+58°	+33°	+55°	+24°	+38°
α (Phillips).	+24	+17	+14	+10	+ 8	+ 8	0	− 2	− 5	− 5	− 7

The improvement in bringing small galactic latitudes to the top of the list is clear. Whatever may be thought of the interpretation, this fact deserves further attention.

36. When we proceed to the interpretation of it we meet some difficulties, which, though not perhaps insuperable, must not be blinked. Before Phillips divided these stars into two groups, the stars with positive α stood at one end of a single series, at the other end of which could be placed our Sun, on certain assumptions. It was suggested that the series depended essentially on the tilt of the axis of rotation to our view. We see the Sun equatorially, and its α is negative; therefore the stars at the other end with α positive must turn towards us one of their poles; and the fact that they lie in low galactic latitudes thus interpreted means that their axes are roughly parallel to the Galaxy.

37. This interpretation can still be applied to the stars of

Group I., for which a changes from about -16 to $+16$ or more. But in Group II., as given by Phillips, there are only 2 stars with a definitely positive. Still a has a range in Group II. comparable with that in Group I., for it starts from a larger negative value -32; and there is nothing incongruous in the supposition that the change from equatorial to polar view which is accompanied in Group I. by a change in l_2 from -16 to $+16$, should be accompanied in Group II. by a change in l_2 from -32 to zero. We should then look for low galactic latitudes at the end of Group II., but we find them on the contrary distinctly *high*: certainly not *low*. If Group II. is correctly formed, the axes of rotation must then be arranged at random—there is no parallelism to the Galaxy as for Group I., which remains unaffected.

38. There is, however, one possible way of bringing Group II. into accordance. It is terminated at the zero end by a detached company of 9 stars. They are shown detached by Phillips's values of ϕ_2, which for these 9 stars are

$$176^\circ \quad 176^\circ \quad 177^\circ \quad 178^\circ \quad 179^\circ \quad 180^\circ \quad 182^\circ \quad 192^\circ \quad 197^\circ,$$

while the 9 stars immediately preceding have, in the reverse order,

$$169^\circ \quad 167^\circ \quad 166^\circ \quad 165^\circ \quad 164^\circ \quad 162^\circ \quad 159^\circ \quad 154^\circ \quad 154^\circ.$$

The gap may of course be purely accidental, but the sequence is so close on either side of it that it attracts attention. If we transfer these detached stars bodily to Group I., as seems to be permissible, then Group II. will also terminate in a series of low galactic latitudes, and we could apply the same reasoning to both groups. The main difficulty is that the values of ϕ_3 for these 9 stars do not fit in very well with Group I. The mean value is 170°, which is at least 20° too small. The values of ϕ_3 agree satisfactorily with Phillips's theoretical values for Group II., except in the case of R Tauri, which is outstanding in either case. If we omit it, the mean ϕ_3 rises to 178°, which is not unreasonably discordant from Group I. There is, however, the further difficulty that the coefficients k_2 for these 9 stars are large—distinctly large for Group I. Thus to make the transference involves us in more than one difficulty. Whether these are fatal can only be settled by further observations.

39. Considering now more particularly Phillips's classification by ϕ_3, it is suggested that the difference between his two groups has reference to minor sinuosities, such as points of inflexion, "pauses" on the rise and fall, or even secondary maxima or minima.

It is suggested that Phillip's Group II. is practically free from such sinuosities, but may have unequal intervals M $-$ m and m $-$ M, and either max. or min. unduly protracted. In attempting to represent these latter features, the second harmonic inevitably introduces sinuosities which are not in the light curve. When

$\phi_2 = 90°$ it introduces a point of inflexion in the rise and fall, and this the third harmonic rectifies if $\phi_3 = 0°$. When $\phi_2 = 180°$ the second harmonic tends to introduce a double "minimum"; and this the third harmonic converts into a merely protracted "minimum," as shown by the light curves if $\phi_3 = 180°$. Hence the relationship between ϕ_3 and ϕ_2.

40. But in Group I. there are actual sinuosities in the light curves. From the study of 5 curves it is suggested that these sinuosities consist essentially of a pause just before max. and another just before min. The third harmonic is chiefly concerned in representing these two pauses, which are a constant feature, and hence tends to a constant value, as Phillips finds.

The Radiative Equilibrium of the Stars. Reply to Mr. Jeans's Criticism. By A. S. Eddington, M.A., M.Sc., F.R.S., Plumian Professor.

In two papers in the *Monthly Notices* for November Mr. Jeans makes some very sweeping criticisms of my theory of the radiative equilibrium of the stars. I cannot admit that his objections have any foundation.

1. In his paper, "The Evolution and Radiation of Gaseous Stars" (p. 37), he refers to my result that, except for a complication arising from ionisation, the radiation of a gaseous star of mass M is of the form KM, where K is a constant. "Professor Eddington nowhere states explicitly the value of K." I contented myself with proving the rather unexpected result that K is a constant; but if Mr. Jeans wishes to know its actual value I can supply it now, since it follows immediately from my formulæ, though not by the process he suggests. $K = \pi cG/k$, where c is the velocity of light, G the constant of gravitation, and k the mass-coefficient of absorption. This is shown by repeating a few steps of my work, supplying the constants which I dropped as unnecessary to the argument. The total radiation is proportional to the surface $4\pi r^2$ and to T_1^4, the actual value being

$$4\pi r^2 \,.\, \tfrac{1}{4} c a T_1^4 \text{ per unit time.}$$

By equation (29) in the first paper, or equation (42) in the second,*

$$a T_1^4 = \frac{g_1}{k}(1 - \beta).$$

Hence, since $g_1 = GM/r^2$, the total radiation is

$$\frac{\pi cG}{k} \,.\, M(1 - \beta).$$

* *Monthly Notices*, vol. lxxvii. pp. 28, 600.

The factor $(1-\beta)$ is the "complication due to ionisation" referred to by Jeans, because when there is no ionisation the value of $(1-\beta)$ is fairly near unity. β is given numerically by my quartic equation (27) or (44), and I have shown that it is independent of the density.

Instead of proceeding in this way, Mr. Jeans reverses the steps of my argument and arrives at the result that the total emission is $4\pi M\epsilon$, which, he remarks, is right but obvious, so that the discussion does not advance our knowledge of stellar conditions in the least. He is not likely to advance our knowledge by undoing my work. He merely verifies my algebra. The unknown quantity ϵ was introduced near the beginning of my paper and eliminated as soon as convenient; the main thing accomplished in my paper was the determination of its value in terms of k, ρ, M, etc. Mr. Jeans throws this aside and resurrects the original symbol.

2. He next states that I obtain an entirely different result in my second paper. He produces no evidence, and I have not the least idea what he is referring to. The formulæ arrived at in the two papers are the same. I stated in § 14 of the second paper that I was merely collecting the theory given in the first paper into a form more convenient for reference and development; I could scarcely have invented a new theory without being aware of it. When it came to translating the formulæ into numbers I adopted a different molecular weight—an improvement largely due to Mr. Jeans's helpful suggestion. It is obvious that his remark cannot refer to this; yet there is certainly no other difference in the results.

3. Mr. Jeans states that it is obvious that there can be no perfectly general laws of this kind for stars supposed to be in a steady state. The answer is that I do not suppose the stars to be in a steady state. Under very general conditions I find the rate at which a giant star of given mass and opacity radiates energy. The star can only be in a steady state if energy from some source is supplied at this rate. If it is difficult to imagine a source supplying energy at just the right rate, that is an argument against the steady state, not against my theory. Jeans's hypothetical star of intense radioactivity will not necessarily radiate faster than an ordinary star; it will simply use the surplus energy in expanding.

4. In his paper, "The Equations of Radiative Transfer" (p. 30), Mr. Jeans believes my equations of radiative transfer to be erroneous. He bases his objection on his equation (10)

$$\frac{3}{7}\frac{dD}{dx}+k\rho C=-2\epsilon\rho,$$

and says that in my papers ϵ is supposed to be finite whilst C and D are neglected as small. "Our exact equation (10) shows this procedure to be illegitimate." On the contrary it exactly justifies it. Taking natural units of length and density, comparable with the radius and mean density of the star, k is of the order 10^{10}.

The equation shows that kC, *i.e.* 10,000,000,000 C, is finite. Surely this verifies my statement that C is negligibly small. I need scarcely say that I have not suggested that we should neglect 10,000,000,000 C.

I have here followed up Jeans's statement as to the relative magnitudes of C and ϵ; but since they are quantities of different physical dimensions, no proper comparison is possible. I suppose that Jeans's point is that in my formulæ I have retained certain terms involving ϵ, and neglected terms involving C; but the legitimacy of this depends on the relative magnitudes of the terms, not on the relative magnitudes of C and ϵ. Accordingly in the last paragraph I chose units which avoid introducing large factors into the terms in question, so that the effects of C and ϵ on my results roughly correspond to their numerical magnitudes.

In my own paper I compared C with B, which is of the same physical dimensions, and I found that C was negligibly small in comparison. By substituting Mr. Jeans's value of ϵ in terms of B in his equation (10) we obtain

$$\frac{3}{7}\frac{d\mathrm{D}}{dx} + k\rho\mathrm{C} = -\frac{2}{3}\frac{d\mathrm{B}}{dx},$$

so that C is of the same order of magnitude as $\frac{1}{k\rho}\frac{d\mathrm{B}}{dx}$.

Now $k\rho dx = 1$ corresponds in an actual star to a length about 1 metre, and the equation shows that C is of the same order as the change of B per metre. This is obviously a very minute fraction of B, except within a few kilometres of the centre where B is negligible as well as C.

5. Considering the importance which Mr. Jeans attaches to this quantity ϵ, it is rather unfortunate that he should use the symbol with a different significance in his paper. In my theory $4\pi\epsilon$ is the net flow of radiation across a spherical surface divided by the mass within that surface *; but Jeans puts $\epsilon = 0$ for the case in which the energy is derived from contraction—which is clearly contrary to my definition. His remarks at the R.A.S. Meeting (*Observatory*, vol. xl. p. 432) seem to imply that the equations developed in his first paper (with $\epsilon = 0$) refer to a star whose energy is generated by contraction. It may be worth while pointing out that this is not so. His result, B = constant, is obtained for a star in a strictly steady state, and is not true when energy is liberated by contraction; so far as that paper goes, the energy must be supplied from a source outside the portion of material under discussion. The fact is that only one of the equations of transfer, $d\mathrm{A}/dx = -k\rho\mathrm{B}$, is really needed; the other is useless (1) because it breaks down when the state is not steady, and (2) because the correct equation of continuity of energy is more easily obtained from first principles. In his second paper, equation (11), Jeans ultimately has to take that procedure, as I have done.

* *Loc. cit.*, p. 20.

The Resonance Theory of the Origin of the Moon.

By Harold Jeffreys, M.A., D.Sc.

I. *Introduction.*

According to the original form of the Nebular Hypothesis of Laplace the history of the moon was not supposed to be notably different from that of any other satellite in the solar system. All were regarded as derived from rings of matter left behind by their primaries when in a gaseous state and contracting on account of loss of heat. An objection that has been urged against this view is that we should expect all satellites to have masses of the same order of magnitude in comparison with their primaries, whereas the moon is relatively much the largest in the system. Its mass is about $\frac{1}{80}$ of that of the earth, while no other satellite gives a ratio of more than $\frac{1}{1000}$. On the Capture Theory no strong correlation would be expected between the masses of satellites and of their primaries, as the chance that a body may be captured is practically independent of its mass. It is uncertain whether there would be appreciable correlation on the Planetesimal Hypothesis, so that on these two theories the large relative mass of the moon is a fact of small significance. When tidal friction is taken into account, on the other hand, this fact is of supreme importance. At present about 82 per cent. of the angular momentum in the terrestrial system arises from the orbital motion of the moon, and only the remaining 18 per cent. from the rotation of the earth. Such a condition is unique in the solar system. Now it is practically certain that angular momentum is being continually transferred from the earth's rotation to the orbital motion of the moon, on account of the friction of the bodily tides within the earth, and that this influence was formerly far more potent than it is now.* On the assumption that sufficient time has elapsed for the changes to take place, which now appears still more probable than it did when the theory was formulated, it was proved by Sir G. H. Darwin that the earth and moon formerly moved very close together, each rotating in such a way as to keep the same face always turned towards the other. The result that the separation was formerly small is confirmed by the fact that the differences between the moments of inertia of the moon correspond with the view that it solidified when its period of revolution was about six of our present days.†

If the former close proximity of the earth and moon be granted, it seems natural to suppose that just before this state was reached the two formed one body. The cause of the separation is however as yet uncertain. It was suggested by Sir G. H. Darwin that the period of rotation of the undivided body was not very different

* Sir G. H. Darwin, *Sci. Papers*, vol. ii.

† *Memoirs R.A.S.*, vol. lx. (1915) pp. 187–217.

from twice the period of one of the free vibrations of a liquid sphere of the same size and density, and that the semi-diurnal solar tide would consequently be very much magnified by resonance, and might reach such a height as to cause rupture of the mass. The theory of the vibrations of a homogeneous rotating spheroid was worked out by Love* and Bryan.† An alternative explanation was afforded by the theory of rotating masses of liquid. An isolated mass of homogeneous liquid at rest is in stable equilibrium in the spherical form. If it is rotating, the free surface is an oblate ellipsoid of revolution (the Maclaurin ellipsoid) unless the angular velocity exceeds a certain value; then the symmetrical form becomes unstable, and the mass takes the form of an ellipsoid with three unequal axes. For still faster rotation this too becomes unstable;‡ for some time it was believed that a pear-shaped figure would then be stable, but Jeans has now shown§ that this is incorrect, and that the mass breaks up at once when such a speed of rotation is reached. This theory has, however, been proved by Moulton‖ to be inapplicable to the case of the earth and moon, for the rotation could never have been fast enough for the Maclaurin ellipsoid to be unstable, so that fission could never have occurred through rotation alone. The only alternatives available at present are to show that heterogeneity or other cause makes the symmetrical form unstable for a much smaller rotation than the homogeneous state requires, to accept the resonance theory, or to abandon the view that the earth and moon formerly formed one body, which is suggested (though not demanded) by tidal theory.

The result of Bryan's investigation was that the condition required by the theory, namely, that resonance should be possible when the disturbing body had a slow positive angular motion relative to axes fixed in direction at the centre of the earth, could only be satisfied if a certain number K_2^2 was small and negative. Now this was also the condition that the Maclaurin ellipsoid should be unstable, which was shown by Moulton to be untrue. Hence resonance was impossible in a homogeneous mass for the principal semi-diurnal tide. It might, however, be possible for tides corresponding to third-harmonic deformations, but the coefficients of these harmonics in the disturbing potential are so small that about 10^{11} vibrations would be necessary before a sufficiently large amplitude could be worked up, and in this time owing to contraction the rotation would have changed enough to destroy resonance. The theory therefore fails for a homogeneous incompressible mass. The object of the present paper is to determine whether hetero-

* *Phil. Mag.*, (5) vol. xxvii. (1889) pp. 254–264.

† *Phil. Trans.*, vol. cliii. *a*. (1889) pp. 187–219.

‡ Lamb, *Hydrodynamics*, pp. 602 *et seq.*, gives an account of the theory, with many references to more detailed investigations.

§ *Phil. Trans.*, vol. ccxvii. *a*. (1916) pp. 1–34.

‖ T. C. Chamberlin and others, *The Tidal and Other Problems*, Carnegie Institute.

geneity will considerably alter this result. The problem in its most general form is too complicated to be completely solved, and the method adopted is to attack a series of soluble problems that will throw light upon the theory.

The first step is to determine the character of the effect of heterogeneity by discussing the analogous problem in two dimensions. As long as the cylindrical mass is homogeneous Bryan's result still holds, and resonance does not occur till instability has set in. It is found, however, that taking account of heterogeneity makes resonance occur for smaller angular velocities, while the mass does not become unstable till it is rotating much more rapidly than the homogeneous body when it is just unstable. Thus the effect is favourable to the theory. Further, when the thickness of the layer of light matter on the outside becomes very small, the angular velocity needed for resonance also becomes small, and, consequently, it can be shown that with any mass, mean density, and angular momentum whatever, a distribution of density can always be found that will give the proper conditions. When we adopt a density distribution that is very like the actual distribution in the earth, one of the free speeds is found to be remarkably near that of the semi-diurnal tide, and it is evident that with a highly probable distribution the agreement would be exact. Thus it seems practically certain that, if the earth and moon ever formed one body, conditions suitable for resonance would have occurred at some stage of the condensation.

It is next shown that the conditions would not change sufficiently rapidly to prevent a large amplitude from being worked up.

Lastly, the course of the changes when resonance occurs in a homogeneous mass is indicated; it appears highly probable that a satellite would be formed by rupture in this case. The problem is however very difficult. It is certain that the body would be very much stretched towards the sun, and that it would remain an exact ellipsoid during the process unless instability arose. Ordinary criteria of stability cannot, however, be applied, for the motion is not even approximately steady. It is most probable that when the elongation was sufficient the body would be longitudinally unstable, just as a liquid cylinder would be, and that it would therefore break up into separate masses. The initial conditions of these would be of just such a character as the theory of the subsequent evolution of the earth and moon requires.

II. *The Free Vibrations of a Heterogeneous Liquid Cylinder.*

In the equilibrium state the liquid is supposed to form two layers, an inner circular cylinder of density $\rho(1+\eta)$ and radius $a_1 = a\alpha$, surrounded by an outer layer of density ρ and radius a. Let the axis of z be along the centre of the cylinder, and suppose a constraint to prevent motion along the cylinder. Let the

angular velocity be ω. The equations of motion in rectangular co-ordinates are

$$\left.\begin{aligned} \ddot{u} - 2\omega\dot{v} - \omega^2 x &= \frac{\partial}{\partial x}\left(\mathrm{V} - \frac{p'}{\rho}\right) \\ \ddot{v} + 2\omega\dot{u} - \omega^2 y &= \frac{\partial}{\partial y}\left(\mathrm{V} - \frac{p'}{\rho}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

where u and v are assumed small; the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad . \quad . \quad . \quad . \quad (2)$$

By cross-differentiation we at once find

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0,$$

so that in all periodic motions

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad . \quad . \quad . \quad . \quad (3)$$

Hence there are a velocity potential Φ and a stream function Ψ giving the motion relative to the rotating axes. Then

$$\nabla^2\Phi = 0 \quad \text{and} \quad \nabla^2\Psi = 0 \quad . \quad . \quad . \quad (4)$$

This will hold for both layers.

Let the equations of the bounding layers in cylindrical polar co-ordinates in the disturbed position be respectively

$$\left.\begin{aligned} r &= a + q\cos(n\phi + \beta) = a + q\mathrm{S}, \text{ say} \\ r &= a_1 + q_1\cos(n\phi + \gamma) = a_1 + q_1\mathrm{S}_1, \text{ say} \end{aligned}\right\} \quad . \quad . \quad (5)$$

where β and γ may be functions of the time.

Then the gravitation potentials in the outer and inner layers are

$$\mathrm{V}_1 = -\pi f\rho(r^2 - a^2) - 2\pi f\rho\eta a_1^2 \log\frac{r}{a_1} + \frac{2\pi f\rho}{n}\left\{aq\mathrm{S}\left(\frac{r}{a}\right)^n + \eta a_1 q_1 \mathrm{S}_1\left(\frac{a_1}{r}\right)^n\right\} \quad (6)$$

$$\mathrm{V}_2 = -\pi f\rho(r^2 - a^2) - \pi f\rho\eta(r^2 - a_1^2) + \frac{2\pi f\rho}{n}\left\{aq\mathrm{S}\left(\frac{r}{a}\right)^n + \eta a_1 q_1 \mathrm{S}_1\left(\frac{r}{a_1}\right)^n\right\} \quad (7)$$

When $r = a$, let the part of $-\dfrac{\partial \mathrm{V}}{\partial r}$ that is independent of ϕ be g; and when $r = a_1$, let the part independent of ϕ be g_1.

We have further

$$V - \frac{p}{\rho} = \int (\dot{u} - 2\omega v - \omega^2 x)dx + (\dot{v} + 2\omega u - \omega^2 y)dy$$
$$= \frac{\partial \Phi}{\partial t} - \tfrac{1}{2}\omega^2 r^2 + 2\omega\Psi \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Assume next that in the inner and outer layers the velocity potentials have the forms

$$\Phi_1 = \left(\frac{r}{a}\right)^n K_1 + \left(\frac{a}{r}\right)^n K_2 \quad . \quad . \quad . \quad (9)$$

$$\Phi_2 = \left(\frac{r}{a_1}\right)^n K_3 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where K_1, K_2, and K_3 are harmonic functions of $n\phi$. Then the stream functions are obtained by writing

$$n\phi - \frac{\pi}{2} \text{ for } n\phi \text{ in } K_1 \text{ and } K_3, \quad \text{and} \quad n\phi + \frac{\pi}{2} \text{ for } n\phi \text{ in } K_2 \quad (11)$$

Then the boundary conditions give

$$n(K_1 - K_2) = a\frac{d}{dt}(qS) \quad . \quad . \quad . \quad . \quad (12)$$

$$n(K_1 a^n - K_2 a^{-n}) = aa\frac{d}{dt}(q_1 S_1) = nK_3 \quad . \quad . \quad (13)$$

Hence

$$K_3 = \frac{aa}{n}\frac{d}{dt}q_1 S_1 \quad . \quad . \quad . \quad (14)$$

$$K_1 = \frac{d}{dt}\frac{a(qS - q_1 S_1 a^{n+1})}{n(1 - a^{2n})}; \quad K_2 = \frac{d}{dt}\frac{a(qSa^{2n} - q_1 S_1 a^{n+1})}{n(1 - a^{2n})}, \quad (15)$$

At the outer surface $\Psi = \frac{d}{dt}\frac{aq}{n}S\left(n\phi - \frac{\pi}{2}\right)$; at the inner $\Psi = \frac{d}{dt}\frac{aa}{n}q_1 S_1\left(n\phi - \frac{\pi}{2}\right)$.

At the outer surface the pressure is constant. Therefore

$$V - \frac{\partial \Phi}{\partial t} + \omega^2 aqS - 2\omega\Psi = \text{constant},$$

or

$$-gqS + \frac{2\pi f\rho a}{n}(qS + \eta a^{n+1} q_1 S_1) + \omega^2 aqS$$
$$- \frac{d^2}{dt^2}\frac{a\{qS(1 + a^{2n}) - 2a^{n+1}q_1 S_1\}}{n(1 - a^{2n})} - \frac{2\omega}{n}\frac{d}{dt}aqS\left(n\phi - \frac{\pi}{2}\right) = 0 \quad (16)$$

At the inner surface the pressure is continuous. Hence

$$\eta\Big[-g_1q_1S_1+\frac{2\pi f\rho a}{n}(qS\alpha^n+\eta q_1S_1\alpha)+\omega^2 a a q_1S_1-2\omega\frac{aa}{n}\frac{d}{dt}q_1S_1\Big(n\phi-\frac{\pi}{2}\Big)\Big]$$
$$-(1+\eta)\frac{aa}{n}\frac{d^2}{dt^2}(q_1S_1)+\frac{d^2}{dt^2}\frac{a\{2qS\alpha^n-q_1S_1a(1+\alpha^{2n})\}}{n(1-\alpha^{2n})}=0 \quad (17)$$

So far nothing has been assumed about the form of the wave, save that S and S_1 are circular functions of the n'th order. Suppose now that the wave travels with its form unchanged and that the speed of the vibration at any point is p. Then q and q_1 are constants, and S and S_1 are of the form $\cos(n\phi-pt+\beta)$.

Then

$$\frac{d}{dt}S\Big(n\phi-\frac{\pi}{2}\Big)=-pS, \text{ and } \frac{dS_1}{dt}\Big(n\phi-\frac{\pi}{2}\Big)=-pS_1 \quad (18)$$

Thus

$$qS\Big[-\frac{ng}{a}+2\pi f\rho+n\omega^2+p^2\frac{1+\alpha^{2n}}{1-\alpha^{2n}}+2\omega p\Big]$$
$$+q_1S_1\Big[2\pi f\rho\eta\alpha^{n+1}-\frac{2p^2\alpha^{n+1}}{(1-\alpha^{2n})}\Big]=0 \quad (19)$$

$$qS\Big[2\pi f\rho\eta\alpha^{n-1}-\frac{2p^2\alpha^{n-1}}{1-\alpha^{2n}}\Big]$$
$$+q_1S_1\Big[-\eta\frac{ng_1}{aa}+2\pi f\rho\eta^2+n\eta\omega^2+(1+\eta)p^2$$
$$+p^2\frac{1+\alpha^{2n}}{1-\alpha^{2n}}+2\eta\omega p\Big]=0 \quad (19)$$

The period equation is therefore

$$\Big(-\frac{ng}{a}+2\pi f\rho+n\omega^2+2\omega p+p^2\frac{1+\alpha^{2n}}{1-\alpha^{2n}}\Big)$$
$$\Big(-\frac{\eta ng_1}{aa}+2\pi f\rho\eta^2+\eta n\omega^2+2\eta\omega p+\eta p^2+\frac{2p^2}{1-\alpha^{2n}}\Big)$$
$$-\Big(\pi f\rho\eta-\frac{p^2}{1-\alpha^{2n}}\Big)^2 4\alpha^{2n}=0 \quad (20)$$

In the homogeneous case $\eta=0$, and this equation reduces to

$$p^2+2\omega p+n\omega^2+2\pi f\rho-ng/a=0,\ p^2=0 \quad . \quad . \quad (21)$$

The zero roots give merely a displacement of the inner boundary without the outer being affected, which is obviously possible when the densities are equal.

In the other type of oscillation the sum of the two possible speeds is -2ω, as was proved by Bryan for the corresponding three-dimensional problem. Further, in the case of the form of bifurcation ($n=2$, $p=0$) $\omega^2=\pi f\rho$, as was found by Jeans.*

We wish to know whether heterogeneity causes resonance or

* "The Equilibrium of Rotating Liquid Cylinders," *Phil. Trans.*, vol. cc. *a.*, 1903, p. 81.

instability to occur for a smaller value of the angular velocity when the mean density is the same. Now the mean density is $\rho(1+\eta a^2)$, and hence the angular velocity that would cause both these conditions in the homogeneous cylinders with the same mean density is given by

$$\omega_1^2 = \pi f \rho (1+\eta a^2) \quad . \quad . \quad . \quad . \quad (22)$$

Resonance will occur in the heterogeneous case if one value of p is -2ω. In this case the angular velocity ω_2 is given by

$$\frac{8\omega^2}{1-a^{2n}}\left\{-\frac{ng}{a}-\frac{\eta n g_1}{a}a^{2n-1}+2\pi f\rho(1+2\eta a^{2n}+2\eta^2 a^{2n})+n\omega^2(1+\eta a^{2n})\right\}$$
$$+\eta\left(n\omega^2+2\pi f\rho-\frac{ng}{a}\right)\left(n\omega^2+2\pi f\rho\eta-\frac{ng_1}{aa}\right)-4\pi^2 f^2\rho^2\eta^2 a^{2n}=0 \quad (23)$$

Instability will commence with an angular velocity ω_3 given by putting $p=0$. Then

$$\left(n\omega^2+2\pi f\rho-\frac{ng}{a}\right)\left(n\omega^2+2\pi f\rho\eta-\frac{ng_1}{aa}\right)-4\pi^2 f^2\rho^2\eta a^{2n}=0 \quad (24)$$

As we are restricted to a two-dimensional problem and only require approximate results to indicate the direction of the effect of heterogeneity, accurate correspondence with the actual case is unnecessary. We shall assume

$$\rho=3{\cdot}2,\ \eta=1{\cdot}50,\ a=0{\cdot}66.\ \text{Then } g/a=3{\cdot}30\pi f\rho;\ g_1/aa=5{\cdot}00\pi f\rho \quad (25)$$

This makes the amounts of matter of the large and small densities in about the same ratio as in Wiechert's hypothesis of the structure of the earth. Then

$$\omega_1^2/\pi f\rho = 1{\cdot}65 \quad . \quad . \quad . \quad . \quad (26)$$

Putting $n=2$ we find

$$\omega_2^2/\pi f\rho = 0{\cdot}71 \quad \text{or} \quad 2{\cdot}10 \quad . \quad . \quad . \quad (27)$$

$$\omega_3^2/\pi f\rho = 2{\cdot}03 \quad \text{or} \quad 3{\cdot}74 \quad . \quad . \quad . \quad (28)$$

If a widely diffused cylindrical mass with the density distributed in this way condenses so as to keep a constant, ρ and ω increase like a^{-2}. Thus $\omega^2/\pi f\rho$ increases like a^{-2}, and resonance will occur when it reaches the value 0·71, which is only 0·43 of the value needed to produce resonance and instability in a homogeneous mass with the same mean density. Thus heterogeneity encourages resonance; the fact that the smaller value of ω_3^2 is greater than ω_1^2 indicates that it discourages instability.

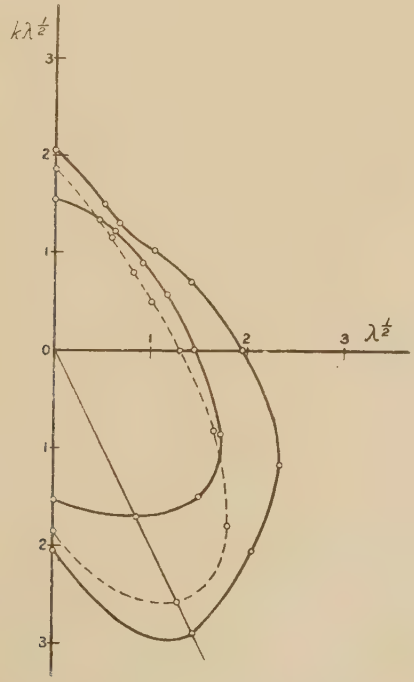
In general, put $p=k\omega$ and $\omega^2/\pi f\rho=\lambda$. Then the period equation for $n=2$ is

$$\left\{-(1+2\eta a^2)+\lambda\left(1+k+\tfrac{1}{2}k^2\frac{1+a^4}{1-a^4}\right)\right\}\times$$
$$\left\{-(2+\eta)+\lambda\left(1+k+\tfrac{1}{2}k^2+\frac{k^2}{\eta(1-a^4)}\right)\right\}\eta-\left\{\eta-\frac{k^2\lambda}{1-a^4}\right\}^2 a^4=0 \quad (29)$$

This may be regarded as a quadratic in λ when k is known. The solution is given in the table. In the last two columns λ_0 is the value of $\omega^2/\pi f\rho$ for a homogeneous cylinder of density $\rho(1+\eta\alpha^2)$, so that a direct comparison is obtained between heterogeneous and homogeneous cylinders of the same mean density.

TABLE I.

k.	λ.		$\lambda^{\frac{1}{2}}$.		$k\lambda^{\frac{1}{2}}$.		$\lambda_0^{\frac{1}{2}}$.	$k\lambda_0^{\frac{1}{2}}$.
∞	0	0	0	0	1·53	2·06	0·00	1·82
3	0·200	0·254	0·447	0·504	1·34	1·51	0·44	1·33
2	0·367	0·423	0·606	0·650	1·21	1·30	0·57	1·15
1	0·83	1·05	0·91	1·03	0·91	1·03	0·81	0·81
0·5	1·30	1·95	1·14	1·40	0·57	0·70	1·01	0·50
0	2·03	3·74	1·42	1·93	0·00	0·00	1·28	0·00
−0·5	2·83	5·43	1·68	2·33	−0·84	−1·12	1·65	−0·83
−1·0	2·34	4·23	1·53	2·06	−1·53	−2·06	1·82	−1·82
−2·0	0·71	2·10	0·84	1·45	−1·68	−2·90	1·28	−2·57
−∞	0	0	0	0	−1·53	−2·06	0·00	−1·82



In the diagram the full curves show the variation of $k\lambda^{\frac{1}{2}}$ with $\lambda^{\frac{1}{2}}$, or for masses with the same densities, of p with ω. The two

branches of the curve approach very closely near $k=2$, but do not intersect. The dotted curve is obtained by plotting $k\lambda_0^{\frac{1}{4}}$ against $\lambda_0^{\frac{1}{4}}$, so that it shows the speeds on the same scale for masses of the same mean density. The condition for resonance is given by the intersection of the curves with the line $p=-2\omega$ or $k=-2$, and it is at once seen that heterogeneity causes this to be satisfied for a much smaller value of the rotation than was otherwise needed. Further, if a be made to approach to unity, two of the values of k can be made to approach zero as closely as we like, whatever λ may be, while the values of λ that make k exactly zero remain finite. Hence by making the depth of the outer layer small enough we can make resonance occur for as small a value of the rotation as we please. It appears physically probable that the same will be true in the three-dimensional problem.

III. *The Original Period of Rotation.*

Wiechert's hypothesis makes the principal moment of inertia of the earth

$$C=\tfrac{8}{15}\pi\rho a^5(1+\eta a^5), \quad . \quad . \quad . \quad (1)$$

and if M be the mass of the earth and m that of the moon

$$C=0{\cdot}334Ma^2 \quad . \quad . \quad . \quad . \quad (2)$$

Let Ω and ω be the respective angular velocities of the earth's rotation and the moon's rotation at present, and let c be the moon's distance. Then the angular momentum of the system is

$$C\Omega+\frac{Mm}{M+m}c^2\omega=5{\cdot}78\ C\Omega \quad . \quad . \quad . \quad (3)$$

Now $\Omega=7{\cdot}29\times10^{-5}/1$ sec. At the same time the moment of inertia of the combined body before separation would be approximately $C(M+m)^{\frac{5}{3}}m^{-\frac{5}{3}}$, apart from the increase due to flattening. Hence the angular velocity when the two formed one body was $5{\cdot}78\ \Omega\left(\frac{M}{M+m}\right)^{\frac{5}{3}}=4{\cdot}14\times10^{-4}/1$ sec, provided that the density distribution was similar, and that the angular momentum and moment of inertia were not different. The only cause of change in angular momentum would be solar tidal friction, which probably would not amount to more than about 5 per cent. of the whole. Thus finally the angular momentum would be enough to make the whole rotate with an angular velocity of about $4{\cdot}3\times10^{-4}/1$ sec, apart from variations in the moment of inertia due to flattening and condensation.

Now a homogeneous ellipsoid would give similarly a maximum angular velocity of $3{\cdot}6\times10^{-4}/1$ sec, and instability in the symmetrical form and resonance would not occur until the rotation speed was $7{\cdot}2\times10^{-4}/1$ sec. Thus the homogeneous mass could never attain conditions suitable for resonance. The effect of heterogeneity on

these conditions seems to be to increase the actual speed, by what has been said above, and to diminish the speed needed for resonance, so that the circumstances are much more favourable.

The problem of finding out the actual amount of the effect of heterogeneity on the period of rotation necessary to cause resonance is likely to be exceedingly difficult, as the bounding surfaces are not ellipsoids. An estimate can, however, be made by analogy with the cylindrical case, which has been accurately solved. In the two-dimensional case considered the angular velocity found necessary for resonance is only 0·65 of that needed when the mass is homogeneous. If the same ratio held in the three-dimensional case, the angular velocity needed would be $4{\cdot}7 \times 10^{-4}/1$ sec, while the available angular velocity is $4{\cdot}1 \times 10^{-4}/1$ sec. There are, however, two causes that will affect the former amount. Compressibility by introducing a further degree of freedom may be expected to reduce the free speed for any given angular velocity, and hence to reduce the angular velocity needed to make the ratio k equal to -2. Further, by reducing the relative thickness of the outer layer this free speed may be indefinitely reduced, and therefore it is evident that there can be a heterogeneous distribution of density, not very different from the actual distribution, such that with the same mean density the available angular momentum would lead to resonance. With still smaller relative thicknesses resonance would still be possible, but would occur at an earlier stage of condensation. The truth of the resonance theory is therefore highly probable.

IV. *The Vibrations of a System with One Degree of Freedom when the Free and Forced Periods are Nearly Equal and Slowly Varying.*

Let the equation of motion of the system be

$$\ddot{x} + \mu\dot{x} + \alpha^2 x = \mathrm{E}e^{ipt} \quad . \quad . \quad . \quad (1)$$

where μ is small, and α, p, and μ are slowly varying.

Put $\int \alpha dt = w$, and let accents denote differentiation with regard to w.

Then

$$x'' + \nu x' + x = \mathrm{E} \exp ip \int \alpha^{-1} dw \quad . \quad . \quad . \quad (2)$$

where

$$\nu = \frac{1}{\alpha}\frac{d\alpha}{dw} + \frac{\mu}{\alpha} \quad . \quad . \quad . \quad . \quad (3)$$

Now let one of the complementary functions of this be $\exp \theta$. Then it has been already shown * that the value of θ is practically $(i - \frac{1}{2}\nu)w$ for a considerable range in w.

Put $x = y \exp \theta$. Then

$$y'' + y'(2\theta' + \nu) = \mathrm{E} \exp\left(ip\int \frac{dw}{\alpha} - \theta\right) \quad . \quad . \quad (4)$$

* *Memoirs R.A.S.*, vol. lx. (1915) pp. 211–213.

If p and α are nearly equal, we can write $p\int\frac{dw}{\alpha}=(1+\beta w)w$, where β is small. Then the expression on the right is $\mathrm{E}\exp(\frac{1}{2}\nu w+\iota\beta w^2)$, which varies slowly. Hence y'' can be neglected on the left, and the particular integral is approximately given by

$$y=\frac{\mathrm{E}}{2\iota}\int\exp(\tfrac{1}{2}\nu w)dw$$

$$=\frac{\mathrm{E}}{\iota\nu}\exp(\tfrac{1}{2}\nu w)$$

near the instant when the periods coincide.

Hence

$$x=\frac{\mathrm{E}}{\iota\nu}\exp(\iota w)+\mathrm{A}\exp(\iota-\tfrac{1}{2}\nu)w+\mathrm{B}\exp-(\iota+\tfrac{1}{2}\nu)w \quad . \qquad (5)$$

where A and B are arbitrary constants. The amplitude of the forced vibration is therefore magnified in the ratio $\alpha\Big/\Big(\frac{d\alpha}{dw}+\mu\Big)$ when the periods coincide.

In the case of a fluid sphere of the same size as the earth μ is small, and hence the magnification is

$$\alpha\Big/\frac{d\alpha}{dw} \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

Now the change of α in one period is $2\pi d\alpha/dw$, and hence the magnification is $2\pi\alpha\div$ the change of α in one period.

In the present case the equilibrium amplitude is about 50 cms., and the amplitude needed to cause rupture is presumably of order 10^8 cms. The condition for this is that the change of α in one period must have been less than $\pi\alpha\times10^{-6}$, so that if α were varying so slowly as not to change greatly in a century, an amplitude of the requisite order of magnitude would be reached when the periods became equal. This hypothesis does not seem unreasonable; it may be considered then that there was time for resonance to produce the required amplitude.

V. *The Movement of a Homogeneous Mass of Rotating Liquid under the Action of a Bodily Force.*

The mass is supposed to remain ellipsoidal. Take the principal axes of the ellipsoid as axes of reference. Let the rotations be ξ, η, ζ, and assume them to be functions of the time alone. Let the angular velocities of the axes relative to fixed axes coinciding with their instantaneous positions be ω_1 ω_2 ω_3.

Then the components of the actual velocity at any point are $-\zeta y+\eta z+\frac{\partial\Phi}{\partial x}$, with two similar expressions, where Φ is a solution

of Laplace's equation. If u, v, w be the velocities relative to the moving axes,

$$u = -(\zeta - \omega_3)y + (\eta - \omega_2)z + \frac{\partial \Phi}{\partial x} \text{ etc.} \quad . \quad . \quad (1)$$

The equation of the free surface is

$$S \equiv 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \quad . \quad . \quad . \quad (2)$$

and the boundary condition is $dS/dt = 0$, where d/dt denotes the total differential following a particle of the fluid. This gives the condition at the free surface

$$\frac{x}{a^2}\frac{\partial \Phi}{\partial x} + \frac{y}{b^2}\frac{\partial \Phi}{\partial y} + \frac{z}{c^2}\frac{\partial \Phi}{\partial z} = \frac{x^2}{a^3}\dot{a} + \frac{y^2}{b^3}\dot{b} + \frac{z^2}{c^3}\dot{c} - (\xi - \omega_1)\frac{b^2 - c^2}{b^2 c^2}yz$$
$$- (\eta - \omega_2)\frac{c^2 - a^2}{c^2 a^2}zx - (\zeta - \omega_3)\frac{a^2 - b^2}{a^2 - b^2}xy \quad (3)$$

From this we can determine Φ, and hence we get

$$u = \frac{\dot{a}}{a}x - (\zeta - \omega_3)\frac{2a^2}{a^2 + b^2}y + (\eta - \omega_2)\frac{2a^2}{a^2 + c^2}z \text{ etc.}, \quad . \quad (4)$$

subject to the equation of continuity

$$\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} = 0. \quad . \quad . \quad . \quad . \quad (5)$$

The equations of motion are three of the form

$$\frac{\partial}{\partial x}\left(V - \frac{p}{\rho}\right) = \frac{du}{dt} - 2\omega_3 v + 2\omega_2 w + \omega_1\omega_2 y + \omega_1\omega_3 z - (\omega_2^2 + \omega_3^2)x - y\dot{\omega}_3 + z\dot{\omega}_2 \quad (6)$$

where V is the total gravitational potential, p the pressure, and ρ the density.

Then

$$\frac{\partial}{\partial x}\left(V - \frac{p}{\rho}\right) = x\left\{\frac{\ddot{a}}{a} - (\zeta - \omega_3)^2\frac{4a^2b^2}{(a^2 + b^2)^2} - (\eta - \omega_2)^2\frac{4a^2c^2}{(a^2 + c^2)^2}\right.$$
$$\left. - 4\omega_3(\zeta - \omega_3)\frac{b^2}{a^2 + b^2} - 4\omega_2(\eta - \omega_2)\frac{c^2}{c^2 + a^2} - \omega_2^2 - \omega_3^2\right\}$$
$$- y\left[\frac{d}{dt}\left\{(\zeta - \omega_3)\frac{2a^2}{a^2 + b^2}\right\} + \frac{2a^2}{a^2 + b^2}\left(\zeta - \omega_3\right)\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right) + 2\omega_3\frac{\dot{b}}{b} + \dot{\omega}_3\right.$$
$$\left. - (\eta - \omega_2)(\xi - \omega_1)\frac{4a^2c^2}{(a^2 + c^2)(b^2 + c^2)} - 4(\xi - \omega_1)\omega_2\frac{c^2}{b^2 + c^2} - \omega_1\omega_2\right]$$
$$+ z\left[\frac{d}{dt}\left\{(\eta - \omega_2)\frac{2a^2}{a^2 + c^2}\right\} + \frac{2a^2}{a^2 + c^2}\left(\eta - \omega_2\right)\left(\frac{\dot{a}}{a} + \frac{\dot{c}}{c}\right) + 2\omega_2\frac{\dot{c}}{c} + \dot{\omega}_2\right.$$
$$\left. + (\zeta - \omega_3)(\xi - \omega_1)\frac{4a^2b^2}{(a^2 + b^2)(b^2 + c^2)} + 4(\xi - \omega_1)\omega_3\frac{b^2}{b^2 + c^2} + \omega_1\omega_3\right] \quad (7)$$

with two similar equations. In order that they may be integrable the coefficient of y in the first must be equal to that of x in the second, with two similar relations. Hence

$$\frac{(b^2+c^2)(c^2+a^2)}{2c^2}\left(\dot\zeta-\zeta\frac{\dot c}{c}\right)-\xi\eta(a^2-b^2)+\omega_1\eta(a^2+c^2)-\omega_2\xi(b^2+c^2)=0 \qquad (8)$$

with two similar equations. If then ξ, η, ω_1, and ω_2 are initially zero, they will remain zero for all time and ζ will be proportional to c.

$V-\frac{p}{\rho}$ is therefore a quadratic function of x, y, and z, save for a possible term independent of the co-ordinates. The coefficient of $\frac{1}{2}x^2$ in it is equal to that of x in (7). The coefficient of xy in it is

$$\frac{2c^2}{(c^2+a^2)(c^2+b^2)}\Big\{(a^2+b^2)\xi\eta-(a^2-c^2)\omega_1\eta-(b^2-c^2)\omega_2\xi + \frac{(c^2+a^2)(c^2+b^2)-4c^2}{2c^2}\omega_1\omega_2\Big\}-(\zeta-\omega_3)\frac{4a^2b^2}{(a^2+b^2)^2}\left(\frac{\dot a}{a}-\frac{\dot b}{b}\right) + \frac{2\omega_3}{a^2+b^2}(a\dot a-b\dot b)-\frac{a^2-b^2}{a^2+b^2}\dot\omega_3 \qquad (9)$$

with two similar expressions.

Now p is zero when

$$S\equiv 1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$

The Laplacian operator acting on $V-\frac{p}{\rho}$ gives

$$-4\pi f\rho-\nabla^2(p/\rho)=\text{a constant}. \qquad (10)$$

Hence

$$\nabla^2 p=\text{a constant}=-2A\rho\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\text{ say}. \qquad (11)$$

Put then

$$p=-A\rho\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)+p' \qquad (12)$$

Then $\nabla^2 p'=0$; and when $S=0$, since p vanishes, we have

$$p'=A\rho \qquad (13)$$

Thus p' is a solution of Laplace's equation, constant over a closed surface and regular within it. It is therefore constant within it, and hence

$$p=A\rho S \qquad (14)$$

Hence V must be of the form V_0+V_2, where V_0 is of zero order and V_2 of the second order in x, y, and z. Put then

$$V_2=-\tfrac{1}{2}\pi f\rho(ax^2+\beta y^2+\gamma z^2+2f'yz+2gzx+2hxy) \qquad (15)$$

Then finally on equating coefficients and eliminating A we have five equations linear in $(\alpha\beta\gamma f' g h)$ and in $(\ddot{a}\ddot{b}\ddot{c}\dot{\omega}_1\dot{\omega}_2\dot{\omega}_3)$. These equations, with $\alpha+\beta+\gamma=4$ are enough to determine the first of these sets of quantities when the second is known. Hence we have the theorem that if a mass of homogeneous liquid has any rotation independent of the position and its free surface is always an exact ellipsoid, the potential function must be a quadratic function of the co-ordinates, however the position and shape of the ellipsoid may be varying. Conversely, these five equations with $\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}=0$ will determine the second set of quantities when the first set and the initial conditions are known. Thus with any possible potential quadratic in the co-ordinates a mass of liquid initially in the form of an ellipsoid moving so that the rotations are independent of the position will remain an ellipsoid moving in this way.

In the case we are concerned with the potential arises mainly from the gravitation of the earth, with a small portion from that of the sun. The orbital velocity of the earth may be neglected, and the sun supposed to be in the plane $z=0$; ξ, η, ω_1, and ω_2 are supposed zero. Let the angle made by the axis of x with the radius vector to the sun be ψ. Then $\dot{\psi}=\omega_3$. The potential due to the earth is

$$\text{Const.} - \pi f\rho_0(a_0x^2+\beta_0y^2+\gamma_0z^2) \quad . \quad . \quad (16)$$

where

$$a_0=\int_0^\infty \frac{d\lambda}{(a^2+\lambda)\mathrm{P}(\lambda)} \quad \text{and} \quad \mathrm{P}^2(\lambda)=(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)$$

with two symmetrical expressions.

That due to the sun is

$$\tfrac{1}{2}\mathrm{E}\{2(x\cos\psi+y\sin\psi)^2-(x\sin\psi-y\cos\psi)^2-z^2\} \quad . \quad (17)$$

where $\mathrm{E}=\pi f\mathrm{M}/r^3$, M being the mass of the sun and r its distance. Consider first cases where the boundary can remain fixed in space, that is to say, $\dot{a}=\dot{b}=\dot{c}=\dot{\omega}=0$. Then we must have $h=0$, by (9), so that either the x or the y axis must pass through the sun. Suppose that the x axis does so. Then the solar part of the potential is $\frac{1}{2}\mathrm{E}(2x^2-y^2-z^2)$, and $\omega=0$. Equating coefficients in the condition that p is constant when $\mathrm{S}=0$, we obtain

$$a^2\left[2\pi f\rho a_0-2\mathrm{E}-\frac{4a^2b^2}{(a^2+b^2)^2}\zeta^2\right]=b^2\left[2\pi f\rho\beta_0+\mathrm{E}-\frac{4a^2b^2}{(a^2+b^2)^2}\zeta^2\right]$$
$$=c^2[2\pi f\rho\gamma_0+\mathrm{E}] \quad (18)$$

These are two equations to determine a, b, and c; they are in general sufficient, as abc is known, being a constant. If E were 0 they would reduce to the equations that determine the form of Maclaurin's and Dedekind's ellipsoids, which are known to give at least one real ellipsoid for any value of ζ. If we put $\sigma=\frac{2ab}{a^2+b^2}\zeta$ the form of this ellipsoid is identical with that of Maclaurin's and

Jacobi's ellipsoids whose angular velocity is σ. The motion is of course quite different, the outer boundary of Dedekind's ellipsoid being fixed in space while the fluid revolves inside it, whereas the whole mass of fluid in Jacobi's ellipsoid rotates like a rigid body. If $a=b$ the value of ω is indeterminate, but the equations (18) still have the same form.

Now let a_0, b_0, c_0 be the semi-axes of the Dedekind ellipsoid corresponding to a given value of ζ, and put $a=a_0+a'$ etc., where $a'b'c'$ are small. The equations (18) reduce to

$$\left(a'\frac{\partial}{\partial a}+b'\frac{\partial}{\partial b}+c'\frac{\partial}{\partial c}\right)2\pi f\rho\{(\alpha_0-\beta_0)a^2b^2+(a^2-b^2)\gamma_0c^2\}=\mathrm{E}\left(3-\frac{c^2(a^2-b^2)}{a^2b^2}\right)$$

$$\left(a'\frac{\partial}{\partial a}+b'\frac{\partial}{\partial b}+c'\frac{\partial}{\partial c}\right)\left\{2\pi f\rho(a^2\alpha_0-b^2\beta_0)-\frac{4a^2b^2(a^2-b^2)}{(a^2+b^2)^2}\zeta^2\right\}=\mathrm{E}(2a^2-b^2) \quad (19)$$

with

$$\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}=0.$$

In general these will determine a' b' and c' uniquely, so that the only effect of the solar disturbance is to make a small change in the form of the mass, thus producing a permanent tide. If, however, the determinant of the coefficients of a' b' and c' vanishes, the values of a' b' and c' will be infinite, as the coefficients of E cannot both vanish. This condition gives the equation

$$\frac{\partial\left[(\alpha_0-\beta_0)a^2b^2-(a^2-b^2)\gamma_0c^2,\ a^2\alpha_0-b^2\beta_0-\dfrac{2a^2b^2(a^2-b^2)}{\pi f\rho(a^2+b^2)^2}\zeta^2,\ abc\right]}{\partial(a,\ b,\ c)}=0 \quad (20)$$

when

$$a=a_0,\ b=b_0,\ c=c_0.$$

If this is satisfied, a steady state near the undisturbed form cannot exist, and $\ddot{a},\ddot{b},\ddot{c},\omega_3$ cannot therefore vanish. In this case the ellipsoid will be slowly drawn out along the axis of x by the disturbing force until it has departed widely from its original form. This corresponds to the case of resonance considered earlier. Equation (9) shows that ψ will cease to be zero, but if the changes are slow it will remain small, so that the longest axis will continue to point nearly to the sun.

The subsequent changes will be more difficult to trace. The departure from the equilibrium form will increase, its *second* derivative remaining roughly constant for a considerable time. The departure will in time become so large that its square and higher powers will need to be taken into account, but this is of little importance from the point of view of the theory. The only difference the terms of higher degree seem likely to cause is that the rotation necessary to make the tidal motion increase indefinitely will have a value slightly different from that indicated by equation

(20). Consequently for a rotation not very different from that found necessary for resonance a distortion comparable with the radius of the body will be produced. It seems probable that the mass will ultimately break up when the extension is great enough. The distorted form will be expected to be very long and narrow, and in such a case the movement of the parts most remote from the centre will be determined more by the attraction of neighbouring matter than by the general gravitative field. Hence there will be a tendency for these to separate from the main mass into one or more detached approximately ellipsoidal nuclei.* The rotation of these will be about equal to that of the primary. The mass velocity of a detached nucleus will be nearly equal to $\frac{2b^2}{a^2+b^2}\zeta x$ parallel to the axis. This is considerably less than ζx, so that the period of revolution will probably be greater than that of the rotation of the primary. The distance apart of the masses will probably be fairly great. The conditions thus indicated are exactly those needed for the subsequent increase of the moon's distance by tidal friction. The criticism of Moulton that the present state of the earth and moon indicates that there could never have been less than about 4000 miles between their surfaces appears to be answered at the same time, for on the resonance theory the initial distance would be expected to be of about this magnitude, or somewhat greater.

Note added December 7th.—A heterogeneous mass may be broken up for another reason, namely, by the formation of a conical point on the surface; immediately after this the surface will become funnel-shaped, and matter will proceed to pour out through the neck formed. The condition for this is that $\partial p/\partial r$ shall vanish at some point on the surface, which reduces practically to $g + v\frac{\partial v}{\partial y} = 0$ at the point just below the sun. A rough calculation has, however, shown that this is not likely to be satisfied for the available rotation.

* This requires the motion to be dynamically unstable at some particular extension; this will probably occur when the longest axis is about three times the shortest. Compare Jeans, *Memoirs, R.A.S.*, vol. lxii. (1917) pp. 15–28. It is certain that the extension actually produced will be of the same order of magnitude as the radius; whether it will actually become as great as this indicates is not yet known.

On the Number of Stars of each Photographic Magnitude in different Galactic Latitudes. By Dr. P. J. van Rhijn.

(*Communicated by Professor J. C. Kapteyn, Assoc. R.A.S.*)

The following is the abstract of a paper which will be published in the *Publications of the Astronomical Laboratory at Groningen*, No. 27. It deals with the number of stars of determined photographic magnitude and galactic latitude between the brightest star occurring in the sky and magnitude 15·5.

The whole sphere has been divided into the following eight galactic zones:—

Zone (5) = zone between galactic latitudes 0 and ± 9.
„ (15) = „ „ „ „ ± 10 and ± 19, etc. etc.
„ (80) = „ „ „ „ ± 70 and ± 90.

The average number of stars in each of these zones has been determined. The magnitude scale on which the investigation has been based is the final Harvard Polar Sequence of *Harvard Annals*, **71**, part 3.

The following data of observation have been used:—

1. *Magnitudes B* to* 5·5.—The photographic magnitudes were derived from the visual ones given in the Revised Harvard Photometry † by applying to the latter a correction ‡ for colour, depending on the spectral type. These counts cover the whole sky.

By a comparison of the photographic magnitudes derived from the visual ones with those of the Göttingen Aktinometrie,§ it appears that the scale of *Harvard Annals*, **50**, corrected for colour, does not deviate very much from the Göttingen scale, which in its turn agrees pretty well with the final Polar Sequence of *Harvard Annals*, **71**.

2. *Magnitudes* 5·5 *to* 7·0.—The stars have been counted in the Göttingen Aktinometrie,§ extending from the equator to + 20° declination. By means of the comparison made by Mr. Chapman ‖ between the scale of *Harvard Circular*, 170, and that of the Göttingen Aktinometrie, it is easily proved that the magnitudes of the latter catalogue are in good agreement with those of *Harvard Annals*, **71**, part 3. Thus the magnitudes of the Göttingen Aktinometrie have been counted off unaltered for each galactic zone.

3. *Magnitudes* 6·5 *to* 8·5.—The magnitudes have been deter-

* B means the magnitude of the brightest star occurring in the sky.

† *Harvard Annals*, **50**, 1908.

‡ *Harvard Annals*, **71**, 7.

§ K. Schwarzschild, "Aktinometrie der Sterne der B.D. bis zur Grösse 7·5 in der Zone 0° bis + 20° Deklination," Teil B. *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen: Mathematisch-Physikalische Klasse*, Neue Folge, Band viii., Nr. 4, 1912.

‖ S. Chapman, "The Determination of Photographic Magnitudes," *The Observatory*, **35**, 267, 1912.

mined at Greenwich with the 6-inch Franklin-Adams lens.* The numbers of stars have been taken from a paper by Chapman and Melotte,† the scale of *Harvard Circular*, 170, used in the Greenwich investigation, being first reduced to the final Polar Sequence of *Harvard Annals*, **71**, part 3. The area counted extends from $+69°$ declination to the north pole, and contains parts of the galactic zones (15), (25), (35), and 45.

4. *Magnitudes* 10·0 *to* 15·5.—The data of observation consist of 65 Northern Durchmusterung plates of the Selected Areas, taken with the Metcalf telescope at Harvard Observatory, and measured at Groningen.‡ The magnitudes have been determined from the estimated diameters of the images, the standards for each area being furnished by Professor Pickering. The latter were observed at Harvard on several photographs of different exposure time. Every Durchmusterung plate contains three images at least of the brighter stars: the main image (exposed sixty minutes), the short-exposure image (exposed one minute), and the prismatic image, formed during the first exposure by a small circular prism attached to the centre of the objective of the telescope. The secondary images have been used for the determination of the magnitudes of the brighter stars, the diameters being standardised by means of Pickering's sequences. Moreover, they served for the derivation of the magnitude corresponding with the diameter estimate 0·0 of the main image, no Harvard standards having been sent to us for these faint stars.

The area measured at Groningen on each of the Durchmusterung plates depends on the galactic latitude as follows:—

TABLE I.

Measured Area on each Durchmusterung Plate. Total Area and Number of Plates counted in different Galactic Zones.

Galactic Latitude.	Measured Area. (Individual Plate.)	Total Area counted.	No. of Plates counted.
$0°$ to $\pm 19°$	$40' \times 40'$	8·89	20
$\pm 20°$ to $\pm 39°$	$60' \times 60'$	17·00	17
$\pm 40°$ to $\pm 90°$	$80' \times 80'$	49·79	28

The following stars have been counted on these areas:—

Stars brighter than magnitude 9·99,
,, between magnitudes 10·00 and 10·49,
,, ,, ,, 10·50 and 10·99, etc. etc.

* F. W. Dyson, "The Photographic Magnitudes of Stars brighter than $9^m·0$ between Declination $+75°$ and the Pole," Greenwich, 1913. "The Photographic Magnitudes of Stars brighter than $9^m·0$ between Declination $+65°$ and Declination $+75°$," Greenwich, 1914.

† S. Chapman and P. J. Melotte, "The Number of Stars of each Photographic Magnitude down to $17^m·0$ in different Galactic Latitudes," *Memoirs of the Royal Astronomical Society*, vol. lx., part 4, 1914.

‡ *Harvard Annals*, **85**.

The limit down to which the stars could be counted was $15^{m}{\cdot}0$ on nearly all plates, and $15^{m}{\cdot}5$ on about half of them. The counting was done in the manuscript of the catalogue.

In order to get the best value for the star-density in each galactic zone, the tabulated numbers were divided by the area counted, and the mean of these quotients was taken for all regions in the same galactic belt. The numbers of plates and the total areas counted have been given in Table 1 for the galactic latitudes 0° to $\pm 19^{\circ}$, $\pm 20^{\circ}$ to $\pm 40^{\circ}$, and $\pm 40^{\circ}$ to $\pm 90^{\circ}$.

Table I. at the end of this paper gives the collected results for the logarithms of the numbers of stars between magnitudes B and *m*. Two values have been inserted in the columns (15), (25), (35), and (45) for the magnitudes 6·49 and 6·99. The upper one refers to the data of the Göttingen Aktinometrie, whereas the lower one has been derived from the Greenwich counts.

The figures of Table II. give the final result for the numbers of stars per square degree brighter than magnitude *m*. They were derived from the data of Table I. by graphical interpolation.

We now pass on to the comparison of the numbers found in the present paper with those derived by two other authorities, viz. Chapman and Melotte * and Kapteyn.† In Table 2 the values

TABLE 2.

Comparison between the Groningen and Greenwich Counts.

Magn. *H.A.* 71, part 3.	Log N_0^{20}.		Log N_{40}^{90}.		Groningen–Greenwich.		$\frac{N_0^{20}}{N_{40}^{90}}$.	
	Gronin-gen.	Green-wich.	Gronin-gen.	Green-wich.	Δ Log N_0^{20}.	Δ Log N_{40}^{90}.	Gronin-gen.	Green-wich.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
5·00	8·57	8·38	8·17	8·05	+0·19	+0·12	2·5	2·1
5·50	8·80	8·66	8·41	8·33	+0·14	+0·08	2·5	2·1
6·00	9·05	8·93	8·65	8·60	+0·12	+0·05	2·5	2·1
7·00	9·52	9·46	9·14	9·13	+0·06	+0·01	2·4	2·2
8·00	0·00	9·97	9·62	9·63	+0·03	−0·01	2·4	2·2
9·00	0·46	0·46	0·08	0·10	0·00	−0·02	2·4	2·3
10·00	0·92	0·91	0·52	0·54	+0·01	−0·02	2·5	2·3
11·00	1·38	1·32	0·92	0·93	+0·06	−0·01	2·9	2·5
12·00	1·80	1·71	1·27	1·30	+0·09	−0·03	3·4	2·6
13·00	2·18	2·06	1·59	1·64	+0·12	−0·05	3·9	2·6
14·00	2·53	2·40	1·88	1·95	+0·13	−0·07	4·5	2·8
15·00	2·86	2·72	2·16	2·25	+0·14	−0·09	5·0	3·0
16·00	3·17	3·01	2·43	2·52	+0·16	−0·09	5·5	3·1

* *Loc. cit.*

† J. C. Kapteyn, "On the Number of Stars of Determined Magnitude and Determined Galactic Latitude," *Publications of the Astronomical Laboratory at Groningen*, No. 18, 1908.

of log N_0^{20} * and log N_{40}^{90},* as computed from the data compiled in this memoir, have been compared with those derived from the Greenwich numbers, the latter being reduced to the scale of *Harvard Annals*, **71**, part 3. † Columns (6) and (7) denote the differences between the Groningen and Greenwich values of log N_0^{20} and log N_{40}^{90} respectively; columns (8) and (9) give the galactic condensation according to both authorities.

The differences between the investigations compared are largest for the very bright and for the very faint stars. It can be shown that the divergence for the bright stars is mainly due to a mistake in the counts of the Göttingen Aktinometrie made at Greenwich.

The most important point relating to the faint stars is the relatively small value found by Chapman and Melotte for the galactic condensation. Now, I think it is easily seen that the methods used at Greenwich for the reduction of the Franklin-Adams counts must lead to values of the galactic condensation, which are systematically too low. For, in order to find the average number of stars per square degree brighter than magnitude m, Chapman and Melotte multiplied the number of stars of each of their gauges in the same galactic zone by some factor, "so as to make the number of 16th magnitude stars (and consequently also the number of stars of other magnitudes in general) approximately the same, say from fifty to eighty, modifying the areas correspondingly. . . . The numbers of stars and the areas were all added together. . . . The mean densities were then computed by dividing by the corresponding areas." ‡

It is clear that we do not find in this way the quantity we are in search of, viz. the number of stars per square degree in any particular galactic zone, in the supposition that all the stars contained in this belt are homogeneously distributed over its area. In order to see this, let us suppose, for argument's sake, that half of the plates in any one zone of determined galactic latitude shows 100 stars per square degree, whereas for the other half this number is only 20.

The mean number of stars per square degree in this case would evidently be $\frac{100+20}{2}=60$. According to Chapman, however, the number 100 in the rich area has to be multiplied by $\frac{1}{5}$ before taking the average; for only then the numbers in the two regions are equal. Thus the Greenwich observers would take the star-density equal to:

$$\frac{\frac{100}{5}+20}{\frac{1}{5}+1}=33\cdot3.$$

* N_0^{20} and N_{40}^{90}, etc. = mean number of stars per square degree from magnitude B to m, for zone between galactic latitudes 0° and ±20°, ±40° and ±90° respectively, etc.

† The differences between the scale of *Harvard Circular*, **170**, used by Chapman and Melotte, and the final North Polar Sequence of *Harvard Annals*, **71**, have been given in *Harvard Annals*, **71**, 224.

‡ S. Chapman and P. J. Melotte, *loc. cit.*, p. 151.

This figure, however, represents the mean number of stars per square degree in a universe where the rich and poor regions occur in the ratio 1 to 5, whereas the real frequency of both areas was supposed to be the same. It thus is clear that the Greenwich value of the star-density is necessarily too small, especially in the galaxy, where the local irregularities in the distribution of the stars are much the largest. I wrote to Chapman about the matter, who kindly re-reduced his numbers, using our plan of reduction; it appears that the revised Greenwich values of the condensation agree pretty well with those found at this laboratory.

In order to compare our results with those derived by Professor Kapteyn,* we have to reduce the photographic scale, on which our investigation depends, to the visual Harvard scale used in *Groningen Publications*, No. 18. This has been done for the bright stars by means of the colour-indices given in the Yerkes Actinometry,† and for the faint ones by means of Seares' observations.‡ After this reduction the differences between the average numbers of stars over the whole sky, as found in the present paper and in *Groningen*, **18**, are small for stars brighter than magnitude ten; for fainter objects the differences increase regularly, until at the 15th magnitude the divergences in log N amount to 0·36, the numbers of *Groningen*, **18**, being the larger ones. It can be shown that these discordances have to be explained by the difference between the visual Harvard scale and the scale given by Parkhurst in his photometric researches,§ from which latter source the standards of the very faintest stars in *Groningen*, **18**, have exclusively been derived. The pretty large differences between the photographic and visual counts have thus to be explained partly by the colour of the faint stars, partly by the deviations of the Yerkes from the Harvard scale. Taking the latter for the correct one, we can easily derive the corrections to be applied to the magnitudes of *Groningen*, **18**, in order to make its numbers agree with those of the present investigation reduced to a visual system. These corrections have been given in Table 3.

TABLE 3.

Corrections to the Magnitudes of Groningen, 18.

Mag. Gron. 18.	True Vis. Mag.	Mag. Gron. 18.	True Vis. Mag.
$<10{\cdot}0$	= Mag. Gron. 18 + 0·00	13·0	= Mag. Gron. 18 + 0·49
$=10{\cdot}0$	= ,, ,, ,, + 0·07	14·0	= ,, ,, ,, + 0·78
11·0	= ,, ,, ,, + 0·14	15·0	= ,, ,, ,, + 1·10
12·0	= ,, ,, ,, + 0·28		

The numbers of stars between the magnitudes B and *m*, as given in *Groningen*, **18**, have been corrected according to these data in Table III. at the end of this paper.

* *Loc. cit.*
† J. A. Parkhurst, *Astrophysical Journal*, **36**, 169, 1912.
‡ *The Astrophysical Journal*, **41**, 206, 1915.
§ J. A. Parkhurst, *Researches in Stellar Photometry*, 1906.

Having thus compared the average numbers of stars over the whole sky, we are now going to make a comparison between both investigations for the galactic and polar regions separately. This has been done for those stars whose average numbers over the whole sphere are the same in both papers.

It appears that the value of the galactic condensation found from the present data agrees very well with that derived in *Groningen Publications*, No. 18; the small excess found from the photographic counts for the bright stars has at least partly to be ascribed to the fact that these bright objects possess on the average a larger actinic power in the Milky Way than at the galactic poles.*

TABLE 4.

Values of the Galactic Condensation $= \frac{N_0^{20}}{N_{40}^{90}}$ *for Bright and Faint Stars.*

Magnitude.		Galactic Condensation.			
Photographic. *H.A.* 71.	Visual. Gron. 18.	This paper Mt. Wilson.	Groningen, 18. Uncorrected.	Groningen, 18. Corrected.	Greenwich.
(1)	(2)	(3)	(4)	(5)	(6)
5·00	4·64	2·5	1·8	2·1	2·1
6·00	5·60	2·5	1·8	2·1	2·1
7·00	6·56	2·4	1·9	2·2	2·2
8·00	7·55	2·4	2·0	2·2	2·2
9·00	8·50	2·4	2·2	2·2	2·3
10·00	9·47	2·5	2·4	2·4	2·3
11·00	10·41	2·9	2·7	2·7	2·5
12·00	11·30	3·4	3·2	3·2	2·6
13·00	12·11	3·9	3·7	3·7	2·6
14·00	12·89	4·5	4·5	4·5	2·8
15·00	13·60	5·0	5·2	5·2	3·0
16·00	14·33	5·5	6·3	6·3	3·1
17·6	16·18	10·2	11·0	11·0	

This table gives the value of the galactic condensation:—

Column (3) according to the present paper for the magnitude in the first column.

Column (4) according to *Groningen*, **18**, for the magnitude in the second column.

Column (5) according to *Groningen*, **18**, corrected for the excess of colour-index of the stars near the galactic poles over those in the Milky Way.

Column (6) according to the Greenwich investigation,† reduced to the scale of *Harvard Annals*, **71**, part 3.

* J. C. Kapteyn, *Introduction to the Cape Photographic Durchmusterung*, 1, (20), 1896. W. de Sitter, *Publications of the Astronomical Laboratory at Groningen*, No. 12, 1904. K. Schwarzschild, *loc. cit.*, p. 35.

† S. Chapman and P. J. Melotte, *loc. cit.*

Each magnitude in the second column has been chosen in such a way that the average number of stars brighter than this magnitude, according to *Groningen*, **18**, is equal to that derived from the present paper for the argument of the first column.

The last line of column (3) gives the value of the condensation computed from Seares' observations;* the corresponding value of the condensation, according to *Groningen*, **18**, has again been computed for the magnitude whose average numbers of stars over the whole sky are the same in Seares' and Kapteyn's investigations. Even for these very faint stars the galactic condensation, as derived from *Groningen*, **18**, agrees pretty well with the Mount Wilson result. We thus see that there exists no such large discordance between the photographic and the visual counts as has sometimes been surmised.

The Greenwich values for the increase of the star-density towards the Milky Way (column (6)) are much smaller than those derived in the present note, especially for the faint stars; but it has been remarked already that the discrepancy disappears if the Greenwich numbers are reduced in the right manner.

TABLE I.

Collected Data for the Logarithms of the Numbers of Stars per Square Degree from B *to Magnitude* m.

Gal. Zone. / *m* (H.S.).	(5)	(15)	(25)	(35)	(45)	(55)	(65)	(80)	
B to 3·99	8·106	8·055	7·775	7·569	7·705	7·701	7·713	7·362	Harvard, 50 + corr. for colour.
,, 4·49	8·328	8·288	8·114	7·818	7·962	7·910	7·972	7·538	
,, 4·99	8·632	8·509	8·364	8·130	8·263	8·135	8·217	7·963	
,, 5·49	8·873	8·736	8·598	8·428	8·456	8·382	8·423	8·362	
,, 5·99	9·122	8·966	8·918	8·716	8·706	8·616	8·675	8·680	Göttingen Aktinometrie. Greenwich Astrographic.
,, 6·49	9·347	9·189 / 9·170	9·151 / 9·067	8·908 / 8·996	8·929 / 8·960	8·867	8·833	8·910	
,, 6·99	9·529	9·381 / 9·455	9·348 / 9·319	9·184 / 9·268	9·166 / 9·198	9·112	9·106	9·118	
,, 7·49	...	9·722	9·582	9·520	9·452	...	...	...	
,, 7·99	...	9·945	9·835	9·755	9·705	...	...	...	
,, 8·49	...	0·140	0·075	9·975	9·952	...	...	...	
,, 8·99									
,, 9·49									

* *Proceedings of the National Academy of Sciences*, 3, 217, 1917.

TABLE I.—*continued.*

Gal. Zone. / (*m* H.S.).	(5)	(15)	(25)	(35)	(45)	(55)	(65)	(80)	
B 9·99	1·041	1·013	0·806	0·591	0·643	0·518	0·505	0·431	Durchmusterung Selected Areas.
,, 10·49	1·215	1·176	1·013	0·869	0·806	0·740	0·633	0·623	
,, 10·99	1·396	1·336	1·182	1·164	1·000	0·919	0·869	0·799	
,, 11·49	1·634	1·571	1·435	1·346	1·210	1·079	1·086	1·000	
,, 11·99	1·829	1·764	1·627	1·508	1·380	1·267	1·253	1·173	
,, 12·49	2·011	1·946	1·797	1·655	1·526	1·393	1·418	1·332	
,, 12·99	2·217	2·157	1·961	1·786	1·673	1·583	1·544	1·519	
,, 13·49	2·374	2·321	2·099	1·932	1·814	1·710	1·688	1·653	
,, 13·99	2·543	2·479	2·253	2·106	1·966	1·855	1·825	1·781	
,, 14·49	2·738	2·653	2·437	2·256	2·130	2·017	1·958	1·915	
,, 14·99	2·918	2·834	2·613	2·398	2·282	2·168	2·097	2·041	
,, 15·49	3·139	2·990	2·766	...	...	2·310	2·258	2·163	

TABLE II.

Final Result. Logarithm of Number of Stars per Square Degree from B *to Magnitude* m.

m (H.S.).	Log N. E=0°.	5°.	10°.	15°.	20°.	30°.	40°.	50°.	60°.	70°.	80°.	90°.
B to 7·00	9·62	9·59	9·52	9·45	9·39	9·29	9·21	9·14	9·12	9·10	9·08	9·08
,, 7·50	9·83	9·81	9·76	9·70	9·64	9·54	9·46	9·39	9·36	9·33	9·32	9·31
,, 8·00	0·07	0·05	9·99	9·93	9·88	9·78	9·70	9·63	9·59	9·56	9·55	9·54
,, 8·50	0·31	0·29	0·23	0·16	0·12	0·02	9·94	9·87	9·82	9·78	9·76	9·75
,, 9·00	0·54	0·51	0·46	0·41	0·36	0·26	0·17	0·09	0·05	0·01	9·99	9·98
,, 9·50	0·77	0·75	0·69	0·64	0·59	0·50	0·41	0·33	0·28	0·23	0·20	0·20
,, 10·00	0·98	0·96	0·91	0·87	0·82	0·72	0·64	0·55	0·49	0·44	0·42	0·40
,, 10·50	1·21	1·19	1·14	1·10	1·04	0·94	0·85	0·75	0·69	0·64	0·61	0·59
,, 11·00	1·43	1·41	1·38	1·33	1·27	1·15	1·05	0·96	0·90	0·84	0·81	0·79
,, 11·50	1·63	1·62	1·59	1·55	1·48	1·35	1·24	1·14	1·08	1·02	0·99	0·97
,, 12·00	1·84	1·83	1·80	1·75	1·68	1·54	1·42	1·30	1·24	1·19	1·16	1·15
,, 12·50	2·03	2·02	2·00	1·95	1·88	1·73	1·60	1·47	1·40	1·36	1·34	1·33
,, 13·00	2·22	2·21	2·19	2·14	2·06	1·88	1·74	1·61	1·56	1·51	1·48	1·47
,, 13·50	2·39	2·38	2·36	2·32	2·22	2·04	1·89	1·76	1·70	1·66	1·63	1·61
,, 14·00	2·56	2·55	2·54	2·49	2·39	2·20	2·03	1·91	1·85	1·80	1·76	1·75
,, 14·50	2·74	2·74	2·70	2·64	2·54	2·34	2·18	2·06	1·98	1·93	1·90	1 88
,, 15·00	2·91	2·90	2·86	2·80	2·70	2·49	2·33	2·21	2·12	2·07	2·03	2·02
,, 15·50	3·06	3·05	3·02	2·98	2·87	2·64	2·47	2·34	2·26	2·19	2·15	2·14
,, 16·00	3·21	3·20	3·17	3·12	3·03	2 78	2·61	2·48	2·39	2·33	2·29	2·27

E = galactic latitude.

Table III.

Logarithm of the Number of Stars per Square Degree from B *to Magn.* m, *according to "Groningen,"* **18**, *corrected for Scale Error.*

	Log N.															
m (H.S.).	E=0°.	5°.	10°.	15°.	20°.	30°.	40°.	50°.	60°.	70°.	80°.	90°.	Log N_0^{20}.	Log N_{20}^{40}.	Log N_{40}^{90}.	Log N_0^{90}.
<10·0.	(See *Groningen Publications*, No. 18.)															
B to 10·0	1·252	1·229	1·157	1·067	0·995	0·875	0·799	0·762	0·733	0·715	0·702	0·702	1·153	0·892	0·748	0·964
,, 10·5	1·486	1·464	1·387	1·292	1·215	1·087	1·004	0·963	0·930	0·907	0·891	0·890	1·383	1·106	0·947	1·184
,, 11·0	1·714	1·690	1·611	1·511	1·428	1·290	1·202	1·155	1·117	1·088	1·069	1·067	1·606	1·311	1·135	1·395
,, 11·5	1·937	1·911	1·828	1·723	1·635	1·488	1·393	1·340	1·295	1·260	1·237	1·234	1·823	1·510	1·316	1·601
,, 12·0	2·149	2·121	2·035	1·924	1·832	1·676	1·574	1·513	1·461	1·420	1·394	1·389	2·030	1·699	1·486	1·797
,, 12·5	2·351	2·321	2·230	2·113	2·018	1·854	1·744	1·676	1·616	1·568	1·538	1·532	2·225	1·879	1·645	1·983
,, 13·0	2·545	2·514	2·418	2·297	2·196	2·026	1·906	1·830	1·762	1·708	1·673	1·665	2·415	2·051	1·795	2·161
,, 13·5	2·737	2·705	2·604	2·478	2·372	2·194	2·065	1·981	1·904	1·843	1·803	1·792	2·601	2·219	1·942	2·338
,, 14·0	2·915	2·881	2·778	2·647	2·536	2·349	2·210	2·118	2·032	1·965	1·919	1·906	2·775	2·375	2·075	2·502
,, 14·5	3·092	3·056	2·950	2·814	2·698	2·501	2·353	2·252	2·157	2·083	2·032	2·016	2·948	2·529	2·205	2·665
,, 15·0	3·266	3·227	3·118	2·978	2·857	2·649	2·493	2·381	2·278	2·195	2·139	2·120	3·116	2·679	2·331	2·824
,, 15·5	3·435	3·395	3·283	3·138	3·012	2·794	2·628	2·506	2·393	$2·301^5$	2·241	2·219	3·281	2·826	2·452	2·980
,, 16·0	3·610	3·569	3·453	3·301	3·170	2·941	2·765	2·633	2·509	2·408	2·342	2·317	3·451	2·976	2·574	3·141

Further Notes on the General Solution of Hill's Equation.

By E. Lindsay Ince, M.A., B.Sc.

§ 1. *Introduction.*—This paper is a sequel to the two papers * I have previously published on the subject of Hill's equation. In the first paper I obtained an expression for the exponent-factor μ in terms of the constants $\Theta_1, \Theta_2, \Theta_3, \ldots$ of the equation and a new parameter σ which itself depends on the constant Θ_0 as well as on $\Theta_1, \Theta_2, \Theta_3, \ldots$. In the second paper I considered the question of the stability or instability of the solutions which might arise when $\Theta_1, \Theta_2, \Theta_3, \ldots$ had the values given in the case studied by Hill, and Θ_0 took all possible values. This is a more or less artificial problem, because, in the astronomical case at least, any variation in Θ_0 implies also variations in $\Theta_1, \Theta_2, \Theta_3, \ldots$. I now propose to consider the case of more interest to astronomers, viz. that in which $\Theta_0, \Theta_1, \Theta_2, \ldots$ are expressed as functions of the quantity m (the ratio of the synodic month to the sidereal year) and vary simultaneously. In the first place I shall obtain an expression for $\cos 2\sigma$ in the form of a series of ascending powers of m, from which I shall deduce a corresponding series for $\sin 2\sigma$. This last I shall make use of to obtain the first six terms of the literal expansion for μ in terms of m, and in conclusion I shall show how $\sin 2\sigma$ varies numerically for a range of values of m from 0 to ·25, and give a rough estimate of the value of m which gives rise to a periodic solution.

§ 2. *The Series for Θ_0 and μ transformed into Series in* m.—As far as terms of the eighth order the expressions which were found for Θ_0 and μ in terms of $\Theta_1, \Theta_2, \Theta_3, \ldots$ and σ are respectively

$$\Theta_0 = 1 - \frac{1}{2^2}\Theta_1^2 - \frac{1}{2\cdot 3}\Theta_2^2 - \frac{11}{2^6\cdot 3}\Theta_1^2\Theta_2 + \frac{1}{2^4\cdot 3}\Theta_1^4 + \cos 2\sigma\left\{\Theta_1 + \frac{1}{2^2}\Theta_1\Theta_2 - \frac{1}{2^6}\Theta_1^3\right\} + \cos 4\sigma\left\{\frac{1}{2^3}\Theta_1^2 + \frac{5}{2^6}\Theta_1^2\Theta_2 - \frac{11}{2^9}\Theta_1^4\right\} \quad . \quad . \quad (1)$$

and

$$\mu = \sin 2\sigma\left\{\frac{1}{2}\Theta_1 + \frac{1}{2^3}\Theta_1\Theta_2 - \frac{3}{2^7}\Theta_1^3\right\} + \sin 4\sigma\left\{\frac{1}{2^7}\Theta_1^2\Theta_2 - \frac{3}{2^{10}}\Theta_1^4\right\} \quad (2)$$

$\Theta_3, \Theta_4, \ldots$ only appear in terms of the tenth and higher orders.

To enable us to express these equations in terms of m, we avail

* *M.N.*, lxxv. p. 436 (1915 March); lxxvi. p. 431 (1916 March).

ourselves of the following relations,* which are correct to the terms in m^8 :—

$$\Theta_0 = 1 + 2m - \frac{1}{2}m^2 + \frac{255}{2^5}m^4 + 19m^5 + \frac{80}{3}m^6 + \frac{533}{2 \cdot 3^2}m^7 + \frac{11230225}{2^{13} \cdot 3^3}m^8,$$

$$\Theta_1 = -\frac{15}{2}m^2 - \frac{57}{2^2}m^3 - 11m^4 - \frac{23}{2 \cdot 3}m^5 - \frac{68803}{2^9 \cdot 3^2}m^6 - \frac{1792417}{2^{10} \cdot 3^3}m^7 - \frac{7172183}{2^7 \cdot 3^4 \cdot 5}m^8,$$

$$\Theta_1^2 = \frac{225}{2^2}m^4 + \frac{855}{2^2}m^5 + \frac{5889}{2^4}m^6 + 371m^7 + \frac{697679}{2^9 \cdot 3}m^8,$$

$$\Theta_1^3 = -\frac{3375}{2^3}m^6 - \frac{38475}{2^4}m^7 - \frac{205605}{2^5}m^8,$$

$$\Theta_1\Theta_2 = -\frac{1665}{2^5}m^6 - \frac{33609}{2^7}m^7 - \frac{169621}{2^8}m^8,$$

$$\Theta_1^4 = \frac{50625}{2^4}m^8,$$

$$\Theta_1^2\Theta_2 = \frac{24975}{2^6}m^8,$$

$$\Theta_2^2 = \frac{12321}{2^8}m^8.$$

With the aid of these relations we are able to express equation (1) in the form

$$2m - \frac{1}{2}m^2 + \frac{465}{2^4}m^4 + \frac{3173}{2^5}m^5 + \frac{63241}{2^7 \cdot 3}m^6$$
$$= \cos 2\sigma \left\{ -\frac{15}{2}m^2 - \frac{57}{2^2}m^3 - 11m^4 - \frac{23}{2 \cdot 3}m^5 - \frac{6023}{2^5 \cdot 3^2}m^6 - \frac{1284239}{2^9 \cdot 3^3}m^7 \right\}$$
$$+ \cos^2 2\sigma \left\{ \frac{225}{2^4}m^4 + \frac{855}{2^4}m^5 + \frac{5889}{2^6}m^6 + \frac{371}{2^2}m^7 + \frac{493983}{2^{12} \cdot 3}m^8 \right\} \quad . \quad (3)$$

while the series in μ takes the form

$$\mu = \sin 2\sigma \left\{ -\frac{15}{2^2}m^2 - \frac{57}{2^3}m^3 - \frac{11}{2}m^4 - \frac{23}{2^2 \cdot 3}m^5 - \frac{18809}{2^9 \cdot 3^2}m^6 - \frac{122707}{2^9 \cdot 3^3}m^7 \right\}$$
$$- \sin 4\sigma \cdot \frac{101925}{2^{14}}m^8 \quad . \quad . \quad . \quad (4)$$

Here only those terms which will be required in the following work have been retained.

* Hill, *Annals of Mathematics*, ix. (1894) pp. 35, 37, and *Collected Works*, iv. pp. 44, 47.

§ 3. *Determination of cos 2σ as a Series of Powers of* m.— Equation (3) shows us that, as a first approximation, cos 2σ has the value $-\frac{2^2}{3\cdot 5m}$. Let us therefore assume a series of the form

$$\cos 2\sigma = -\frac{2^2}{3\cdot 5m}[1 + a_1 m + a_2 m^2 + a_3 m^3 + a_4 m^4 + a_5 m^5 + \ldots],$$

whence also

$$\cos^2 2\sigma = \frac{2^4}{3^2\cdot 5^2 m^2}[1 + 2a_1 m + (a_1^2 + 2a_2)m^2 + (2a_1 a_2 + 2a_3)m^3 + \ldots].$$

The coefficients $a_1, a_2, a_3, \ldots$ have now to be determined, we therefore substitute these assumed values of cos 2σ and $\cos^2 2\sigma$ in equation (3) and compare the coefficients of successive powers of m on the two sides of the equation obtained. We find that the coefficient of m gives rise to

$$2 = \left(-\frac{15}{2}\right)\left(-\frac{2^2}{3\cdot 5}\right),$$

an identity. Taking the coefficients of m^2, we find that

$$-\tfrac{1}{2} = 2a_1 + \frac{57}{3\cdot 5} + 1,$$

whence

$$a_1 = -\frac{53}{2^2\cdot 5}.$$

The coefficients of m^3 give rise to the equation

$$0 = 2a_2 + \frac{57}{3\cdot 5}a_1 + \frac{44}{3\cdot 5} + 2a_1 + \frac{19}{5},$$

whence

$$a_2 = \frac{2591}{2^3\cdot 3\cdot 5^2}.$$

Thus in succession we find that

$$a_3 = \frac{156073}{2^4\cdot 3^2\cdot 5^3},$$

$$a_4 = \frac{13328113}{2^6\cdot 3^3\cdot 5^4},$$

and

$$a_5 = \frac{2054674331}{2^9\cdot 3^4\cdot 5^5}.$$

The series for cos 2σ therefore is

$$\cos 2\sigma = -\frac{2^2}{3\cdot 5m}\left[1 - \frac{53}{2^2\cdot 5}m + \frac{2591}{2^3\cdot 3\cdot 5^2}m^2 + \frac{156073}{2^4\cdot 3^2\cdot 5^3}m^3 + \frac{13328113}{2^6\cdot 3^3\cdot 5^4}m^4 + \frac{2054674331}{2^9\cdot 3^4\cdot 5^5}m^5 + \ldots\right] \quad . \quad (5)$$

§ 4. *The Series for sin* 2σ.—The next step is to obtain a similar series for sin 2σ. Let us assume therefore the expansion

$$\sin 2\sigma = \pm \frac{2^2 i}{3 \cdot 5\mathrm{m}}\left[1 + b_1\mathrm{m} + b_2\mathrm{m}^2 + b_3\mathrm{m}^3 + b_4\mathrm{m}^4 + b_5\mathrm{m}^5 + \ldots\right], \qquad (i = \sqrt{-1})$$

whence

$$\sin^2 2\sigma = -\frac{2^4}{3^2 \cdot 5^2\mathrm{m}^2}\left[1 + 2b_1\mathrm{m} + (b_1^2 + 2b_2)\mathrm{m}^2 + (2b_1b_2 + 2b_3)\mathrm{m}^3 + \ldots\right],$$

and use the identity

$$\cos^2 2\sigma + \sin^2 2\sigma = 1$$

to set up the relations between the a's and the b's.

From the coefficients of $\frac{1}{\mathrm{m}}$ in this identity we have

$$0 = a_1 - b_1,$$

whence

$$b_1 = -\frac{53}{2^2 \cdot 5}.$$

From the coefficients of m^0 we have

$$\frac{3^2 \cdot 5^2}{2^5} = a_2 - b_2 + \tfrac{1}{2}(a_1^2 - b_1^2),$$

whence

$$b_2 = -\frac{6511}{2^5 \cdot 3 \cdot 5^2}.$$

Likewise from the coefficients of m we have

$$0 = a_1a_2 + a_3 - b_1b_2 - b_3,$$

whence

$$b_3 = -\frac{1434541}{2^7 \cdot 3^2 \cdot 5^3},$$

and in succession from the coefficients of m^2 and m^3 we obtain

$$b_4 = -\frac{1084909759}{2^{11} \cdot 3^3 \cdot 5^4}$$

and

$$b_5 = -\frac{1858391325 79}{2^{13} \cdot 3^4 \cdot 5^5}.$$

Thus we obtain a series expression for sin 2σ, viz.

$$\sin 2\sigma = \pm \frac{2^2 i}{3 \cdot 5\mathrm{m}}\left[1 - \frac{53}{2^2 \cdot 5}\mathrm{m} - \frac{6511}{2^5 \cdot 3 \cdot 5^2}\mathrm{m}^2 - \frac{1434541}{2^7 \cdot 3^2 \cdot 5^3}\mathrm{m}^3 - \frac{1084909759}{2^{11} \cdot 3^3 \cdot 5^4}\mathrm{m}^4 - \frac{1858391325 79}{2^{13} \cdot 3^4 \cdot 5^5}\mathrm{m}^5\right] \qquad (6)$$

We also have

$$\sin 4\sigma = 2 \sin 2\sigma \cos 2\sigma = \pm \frac{2^5 i}{3^2 \cdot 5^2 m^2}\left[1 - \frac{53}{2 \cdot 5}m - \ldots\right].$$

The sign of these expressions is of course ambiguous; we shall arbitrarily choose the positive sign.

§ 5. *The Literal Expression for* c.—The expression (4) which gives μ in terms of sin 2σ and m may be written in the form

$$i\mu = i \sin 2\sigma \left\{ -\frac{15}{2^2}m^2 - \frac{57}{2^3}m^3 - \frac{11}{2}m^4 - \frac{23}{2^3 \cdot 3}m^5 - \frac{18809}{2^9 \cdot 3^2}m^6 - \frac{122707}{2^9 \cdot 3^3}m^7 - \ldots \right\}$$

$$+ i \sin 4\sigma \left\{ -\frac{101925}{2^{14}}m^8 - \ldots \right\}.$$

If we use the series for sin 2σ and sin 4σ in terms of m, obtained in the preceding section, we may express μ as a series of ascending powers of m alone. We find, in fact, that on the right-hand side of the equation

the coefficient of $m = +1$,

" " $m^2 = b_1 + \frac{19}{2 \cdot 5} = -\frac{3}{2^2}$,

" " $m^3 = b_2 + \frac{19}{2 \cdot 5}b_1 + \frac{22}{3 \cdot 5} = -\frac{201}{2^5}$,

" " $m^4 = -\frac{2367}{2^7}$,

" " $m^5 = -\frac{111749}{2^{11}}$,

" " $m^6 = -\frac{4095991}{2^{13} \cdot 3}$.

The exponent c being defined by the relation

$$c = 1 + i\mu,$$

we thus arrive at the known literal expression for c, viz.

$$c = 1 + m - \frac{3}{2^2}m^2 - \frac{201}{2^5}m^3 - \frac{2367}{2^7}m^4 - \frac{111749}{2^{11}}m^5 - \frac{4095991}{2^{13} \cdot 3}m^6 - \ldots \quad (7)$$

§ 6. *Numerical Results—a Periodic Solution.*—In my previous papers on the subject of Hill's equation I pointed out that when $\sin 2\sigma$ vanishes, μ vanishes, and the solution of Hill's equation becomes purely periodic; when $\sin 2\sigma$ is purely imaginary, μ also is purely imaginary and the solution is stable. It is therefore interesting to examine how $\sin 2\sigma$ varies as m assumes different values in a convenient range.

The numerical values of the coefficients b in the expansion of $\sin 2\sigma$ are as follows:—

$$b_1 = -\frac{53}{20} = -2{\cdot}65000,$$

$$b_2 = -\frac{6511}{2400} = -2{\cdot}71292,$$

$$b_3 = -\frac{1434541}{144000} = -9{\cdot}96209,$$

$$b_4 = -\frac{1084909759}{34560000} = -31{\cdot}39206,$$

$$b_5 = -\frac{185839132579}{2073600000} = -89{\cdot}62150.$$

The series $1 + b_1 m + b_2 m^2 + b_3 m^3 + \ldots$ is therefore not very convergent for values of m greater than about 0·1. The ratios $\frac{b_3}{b_2}$, $\frac{b_4}{b_3}$, and $\frac{b_5}{b_4}$ are respectively 3·67, 3·15 and 2·85. On the assumptions that the b's always remain negative (so that the series has one zero) and that the series tends to become geometric, we may fairly assume the radius of convergence to be not less than ·4. If this is so we may take the remainder after the term in m^5 to be of the form

$$225 m^6 [1 + 2{\cdot}5m + (2{\cdot}5m)^2 + (2{\cdot}5m)^3 + \ldots]$$

or

$$\frac{225 m^6}{1 - 2{\cdot}5m}.$$

In the following table, the column S_6 gives the values of $\sin 2\sigma$ when the first six terms of the series only are retained. The numbers in the column headed S take into account also the above assumed value for the remainder of the series. The last column gives the corresponding value of $i\mu$. As the series for $i\mu$ appears to diverge when m is at all greater than 0·2, little reliance can be put on the last entries in this column.

The series for sin 2σ seems to be more satisfactory than the series for $i\mu$ for investigating the question of periodicity.

m.	S_6.	S.	$i\mu$.
·00	∞i	∞i	0
·05	$4{\cdot}5828i$	$4{\cdot}5828i$	·0472
·08085	$2{\cdot}5104i$	$2{\cdot}5101i$	·0716
·10	$1{\cdot}8503i$	$1{\cdot}8495i$	·0835
·15	$\cdot8625i$	$\cdot8586i$	·107
·20	$\cdot2705i$	$\cdot2323i$	·04
·21	$\cdot1700i$	$\cdot1176i$	·02
·22	$\cdot0726i$	$\cdot0042i$	·00
·23	$-\cdot0228i$	$-\cdot1133i$	...
·24	$-\cdot1173i$	$-\cdot2362i$	...
·25	$-\cdot2111i$	$-\cdot3666i$	...

$m = \cdot08085$ corresponds to the case of the lunar perigee discussed by Hill.

If, as we have supposed, all the coefficients b in the series $1 + b_1 m + b_2 m^2 + \ldots$ are negative, the series must have one zero for some positive value of m. For this value of m, sin 2σ vanishes, and consequently μ vanishes and the solution is purely periodic. The above table indicates that this zero of sin 2σ occurs at a value of m near 0·22. In other words, for values of m about $\frac{2}{9}$, the apse of the orbit considered has only a vanishingly small motion.

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On the Errors in a Sum of Tabular Quantities.

By H. C. Plummer, M.A.

1. The law of errors to be expected in a sum of tabular quantities presents a problem which has been proposed by Professor E. W. Brown, and discussed by Dr. Schlesinger in a very interesting article.*

A practical answer to the question needs no very recondite consideration, and is probably quite familiar. The limiting errors are $\pm\frac{1}{2}$ in terms of one unit in the last decimal place, and the frequency is constant for all errors between these limits. The probable error is $\pm\frac{1}{4}$, and the error of mean square for n direct tabular numbers is evidently $\frac{1}{2}\sqrt{(n/3)}$. When n is large the distribution for the errors of the sum naturally approaches the Gaussian type, and the probable error of the sum approaches a constant ratio to the error of mean square. This ratio is 0·6745 . . .,

* *Astronomical Journal*, No. 717.

but for all practical purposes may be replaced by 2/3. The probable error of the sum of any considerable number n of direct tabular entries is therefore $\pm\frac{1}{3}\sqrt{(n/3)}$ or $\pm\sqrt{(n/27)}$. But when n is only 1 the probable error is $\pm\frac{1}{4}$. Hence for moderate values of n the probable error is intermediate between $\pm\cdot25\sqrt{n}$ and $\pm\cdot192\sqrt{n}$. As a matter of fact Dr. Schlesinger shows that the latter value is practically reached already when $n=2$. Thus in the sum of 27 items taken directly from a table, the probable error is ±1. But while the error of the sum is equally likely to fall short of or to exceed one unit in the last tabular place, errors of 3 or 4 units will not be infrequent. The term "probable error" is one of the most unfortunate ever introduced into exact science.

2. In a second problem discussed by Dr. Schlesinger the items of the sum are obtained by linear interpolation from the table with perfect accuracy, or, in other words, without restricting the results to the tabular number of decimal places. The interpolating fraction being z and the tabular errors being e_1, e_2, the square of the error of mean square is

$$\Sigma\{ze_1+(1-z)e_2\}^2=\tfrac{1}{12}\Sigma\{2z^2+2(\tfrac{1}{2}-z)\}=\tfrac{1}{12}\cdot\tfrac{2}{3}n=\tfrac{1}{18}n,$$

when n is large and z has all values equally between 0 and 1. The error of mean square for a sum of n such interpolated items is therefore $\frac{1}{3}\sqrt{(n/2)}$, and the probable error when the distribution becomes Gaussian is $\pm\sqrt{(2n)}/9$ or $\pm\sqrt{(n/40)}$. Thus a probable error of one tabular unit is obtained from the sum of 40 such interpolations, as compared with the corresponding sum of 27 direct tabular numbers.

If, however, the interpolations are reduced to the tabular limit before the sum is taken an independent error is added, and for it the magnitude of the mean square is $1/\sqrt{12}$. Hence the mean square error of the sum is $\sqrt{\{n(\frac{1}{12}+\frac{1}{18})\}}$ or $\sqrt{(5n)}/6$, and the probable error approaches $\pm\sqrt{(5n)}/9$ or $\pm\sqrt{(n/16)}$. On the same standard of a probable error of one tabular unit, a sum of 16 of these items is on the same footing as a sum of 27 direct tabular numbers or a sum of 40 ideal (unrounded) linear interpolations.

This is the form in which Dr. Schlesinger expresses his more striking practical deductions. He is probably right in thinking that a higher accuracy is indicated than would generally be supposed without a little reflection. But it may be possible to press this aspect too far. If the final figure of the table is suppressed, the entries will have a probable error of $\pm2\cdot5$ in the previous unit. But there is a definite limit of 5 units in the error. With the sums of 27, 40, or 16 items defined above there is a probable error of 1 unit only, but even 5 units is not a definite limit to the possible error. On the whole a rough rule, that a sum of 20 to 40 terms, derived from tables without excessive precautions, is practically equivalent to the loss of the last tabulated place, seems a fairly safe and convenient guide.

3. The practical interest of the question depends on n being large,

in which case the distribution certainly approaches the Gaussian type. But Dr. Schlesinger's examination of the frequency in cases when n is small has considerable theoretical interest, and some comment from another point of view may be allowed.

For a single tabular quantity the frequency of an error between e and $e+de$ is $1 . de$ between $\pm\frac{1}{2}$ and 0 outside these limits. For the sum of two such quantities consider a square, whose sides are $x=\pm\frac{1}{2}$, $y=\pm\frac{1}{2}$. The errors of two tabular numbers may be represented by the coordinates of any point within the square. The line $x+y=e$ within the square is the locus of all the points which give the error e for the sum, and the area between this line and the parallel diagonal represents the probability that the error of the sum lies between 0 and e. Let e be positive. The length of the perpendicular from the centre of the square is $e/\sqrt{2}$, and the area required is $\frac{1}{2}\{1-(1-e)^2\}=e-\frac{1}{2}e^2$. Hence the frequency of positive errors between e and $e+de$ is clearly $(1-e)de$, for this is the element of area which representative points in the square may occupy. The discontinuity between positive and negative errors is geometrically obvious, but the frequency may always be expressed by $(1-|e|)de$. The ordinates of the frequency curve are represented by the lengths of parallel lines in the square.

The probable error of a sum of two items is found by drawing a line parallel to the diagonal in such a way as to cut off a quarter of the square. The intercepts on the sides are $1/\sqrt{2}$, and the equation of the line is $x+y=\frac{1}{2}-1/\sqrt{2}+\frac{1}{2}$. The probable error is therefore $1-1/\sqrt{2}$ or $(1/\sqrt{2}-\frac{1}{2})\sqrt{n}$, since $n=2$. This is $0{\cdot}207\sqrt{n}$, already a close approximation to the probable error which would follow from the error of mean square according to the Gaussian distribution.

4. Quite similarly, when there are three items in the sum, the ordinates of the frequency curve are represented by the areas of the plane sections $x+y+z=e$ of the cube whose sides are $x=\pm\frac{1}{2}$, $y=\pm\frac{1}{2}$, $z=\pm\frac{1}{2}$. If the plane be imagined to move from the point $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, the sequence of changes and discontinuities will at once be seen. The section is an equilateral triangle until it passes through the points $(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$, $(\frac{1}{2},-\frac{1}{2},\frac{1}{2})$, $(-\frac{1}{2},\frac{1}{2},\frac{1}{2})$. After this it cuts six sides instead of three until, passing over the centre of the cube, it goes through the points $(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$, $(-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$, $(\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$, and ends with a series of equilateral triangles approaching the point $(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$. To see this succession of changes clearly is a good simple test of geometrical imagination.

At first the plane intersects three sides of the cube in $(\frac{1}{2},\frac{1}{2},e-1)$, $(\frac{1}{2},e-1,\frac{1}{2})$, $(e-1,\frac{1}{2},\frac{1}{2})$, so long as $\frac{1}{2}<e<\frac{3}{2}$. The side of the triangle is $\sqrt{2}(\frac{3}{2}-e)$, and its area is therefore $\frac{1}{2}\sqrt{3}(\frac{3}{2}-e)^2$. The central perpendicular on the plane is $e/\sqrt{3}$. Hence the element of volume which measures the frequency between e and $e+de$ is $\frac{1}{2}(\frac{3}{2}-e)^2de$.

In the central part the section is a hexagon. The intersections with the sides of the cube are in order $(-\frac{1}{2},\frac{1}{2},e)$, $(-\frac{1}{2},e,\frac{1}{2})$, $(e,-\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},-\frac{1}{2},e)$, $(\frac{1}{2},e,-\frac{1}{2})$, $(e,\frac{1}{2},-\frac{1}{2})$. The lengths of the

sides of the hexagon are alternately $\sqrt{2}(\frac{1}{2}-e)$ and $\sqrt{2}(\frac{1}{2}+e)$, and they are parallel to the sides of a regular hexagon, which is the figure of the central section. The area is therefore equal to the area of an equilateral triangle with sides $\sqrt{2}(\frac{3}{2}-e)$, *less* 3 equilateral triangles with sides $\sqrt{2}(\frac{1}{2}-e)$. Hence in the interval $-\frac{1}{2}<e<\frac{1}{2}$ the area of the section is $\frac{1}{2}\sqrt{3}\{(\frac{3}{2}-e)^2-3(\frac{1}{2}-e)^2\}$, and the frequency between e and $e+de$ is represented by the element of volume

$$\tfrac{1}{2}\{(\tfrac{3}{2}-e)^2-3(\tfrac{1}{2}-e)^2\}de=(\tfrac{3}{4}-e^2)de.$$

The reasoning supposes e positive, but the result shows that there is no discontinuity at the centre, and it is unnecessary to repeat the argument for the case of e negative. The unreduced form, with its mode of derivation, will presently be seen to be the more interesting. The third part of the cube, $-\frac{3}{2}<e<-\frac{1}{2}$, repeats the features of the first part, and the same formula applies if e be replaced by $|e|$, giving $\frac{1}{2}(\frac{3}{2}-|e|)^2$.

The problem of finding the probable error of the sum of three tabular numbers is equivalent to finding the plane $x+y+z=e$ which cuts off a quarter of the cube. It clearly lies in the central part, and is therefore given by

$$\tfrac{1}{4}=\tfrac{3}{4}e-\tfrac{1}{3}e^3.$$

The root is 0·353 or $0\cdot204\sqrt{3}$.

5. As a mode of calculation this method is not suitable for extension to higher values of n. But the analogous geometrical interpretation of the results to be expected is certainly curious. The series clearly goes on to the $(n-1)$-dimensional sections of the n-dimensional analogue of the square and cube. These sections are represented by n polynomials of degree $(n-1)$, each covering unit range in e and forming a function with singularities at the $(n-1)$ points of junction, or $(n+1)$ points if the two ends of the range be included, where the function becomes zero. The series of such functions, each represented by a number of polynomials in different parts of the range, tend to approach the normal probability function, and this fact seems to constitute a very singular proposition in the geometry of hyperspace.

Dr. Schlesinger has continued the calculation as far as $n=6$ by his method of successive derivation. It seems desirable to find the general form of the function for any value of n. At first sight the expressions found are far from promising to any attempt to obtain the general result by induction. But the idea of single discontinuous functions for each value of n soon provides the clue. It is only necessary to consider the nature of the singularities. When $n=1$ the function drops from 1 to 0 at both ends of the range, and there are two discontinuities in the value of the function itself. When $n=2$ the function is represented by two straight lines, and there are three discontinuities in the first derivative. When $n=3$ the function is represented by three parabolas. There is no discontinuity in the direction of the tangent, but there are

four discontinuities in the second derivative. At the end of the series, the probability function, the derivatives are all continuous. Hence it is natural to suspect that the singularities occur always as discontinuities in the $(n-1)$th derivative. But the function on the two sides of a point of junction is represented by polynomials of degree $(n-1)$. Therefore they can only differ by the $(n-1)$th power of the distance from the point of junction. This is easily verified in Dr. Schlesinger's results. But the end polynomial is

$$\int_{-\frac{1}{2}}^{\frac{1}{2}(n-1)} de \ . \ . \ . \int_{e-\frac{1}{2}}^{1} de \int_{e-\frac{1}{2}}^{\frac{1}{2}} de = \frac{1}{(n-1)!}(\tfrac{1}{2}n-e)^{n-1}.$$

Hence the complete law will really be represented in the central section of the range, where it is a sum of similar binomial forms to the power $(n-1)$. It only remains to discover the law of the coefficients, and these are easily found to be binomial coefficients of the expansion $(1-x)^n$. Thus finally we arrive at the general law for the frequency of errors between e and $e+de$ in sums of n tabular items, namely,

$$\frac{de}{(n-1)!}\Big\{(\tfrac{1}{2}n-|e|)^{n-1}-n(\tfrac{1}{2}n-1-|e|)^{n-1} + \frac{n(n-1)}{2!}(\tfrac{1}{2}n-2-|e|)^{n-1}- \ . \ . \ .\Big\}$$

where the expansion is to be continued until the contents of any bracket become negative. The greatest number of terms is $\frac{1}{2}n$ or $\frac{1}{2}(n+1)$ according as n is even or odd. These are all represented in the range containing or adjacent to 0, and in the successive sections of length 1 in e the terms drop off one by one from the right until the first term alone is left at the end of the range.

It will now be noticed how the second term is represented for the case $n=3$ by the truncated corners of the triangular sections of a cube after the plane has passed the position containing three corners of the cube.

6. The general expression for the law of frequency, once obtained, can now be proved by induction without difficulty, with the help of Dr. Schlesinger's method of successive integration.

But a more direct proof can be given. It is more convenient to allow the individual errors to range from 0 to 1. The probability of errors between e_1+de_1, e_2+de_2, . . . is de_1de_2 . . . Hence the frequency of an error in the sum between 0 and E is

$$\iiint \ . \ . \ . \ de_1, de_2, de_3 \ . \ . \ .$$

subject to the condition $e_1+e_2+e_3+ \ . \ . \ . \geqslant \mathrm{E}$, but with the further condition that $e_1, e_2, \ . \ . \ .$ cannot exceed 1. This is the simplest case of the Lejeune-Dirichlet integral, and its value according to the first condition only is

$$\mathrm{E}^n/n\,!$$

For the range $0<E<1$, the further condition is necessarily satisfied. But in the range $1<E<2$, some of the combinations contain $e_1>1$, or $e_2>1$, . . . Their effect must be excluded by adding

$$-n\,.\,(E-1)^n/n\,!$$

for each such combination is matched by one in the previous range, from which it may be derived by adding 1 to any one of the n errors e_1, e_2, . . .

If the sum of these two terms is applied in the range $2<E<3$, all those combinations in which e_1 exceeds 1 and e_2 exceeds 1 have been independently rejected. For all those which result from adding 1 to any item in the range 0 to $E-1$ have been excluded from the total. But this means that those which result from adding 1 to both e_1 and e_2 in combinations in the range 0 to $E-2$ have been subtracted twice. They must be restored for each pair e_1, e_2 by adding

$$+\tfrac{1}{2}n(n-1)\,.\,(E-2)^n/n\,!$$

It is unnecessary to pursue an argument of familiar type further. It follows that the frequency between 0 and E is represented by

$$\frac{1}{n\,!}\left\{E^n-n(E-1)^n+\binom{n}{2}(E-2)^n-\binom{n}{3}(E-3)^n+\ \ldots\right\}$$

and that the frequency between E and $E+dE$ is

$$\frac{dE}{(n-1)\,!}\left\{E^{n-1}-n(E-1)^{n-1}+\binom{n}{2}(E-2)^{n-1}-\ \ldots\right\}$$

The range may be changed to $\pm\frac{1}{2}$, or $\pm\frac{1}{2}n$ in the sum, by writing $e+\frac{1}{2}n$ for E, and then the frequency is

$$\frac{de}{(n-1)\,!}\left\{(\tfrac{1}{2}n+e)^{n-1}-n(\tfrac{1}{2}n-1+e)^{n-1}+\binom{n}{2}(\tfrac{1}{2}n-2+e)^{n-1}-\ \ldots\right\}$$

the series continuing until the contents of a bracket become negative. Formally this differs from the previous result, which is obtained by writing $-|e|$ for e. The symmetry which results from the fact that every combination is matched by a combination of complementary or opposite errors permits the change, and makes for economy in the number of terms. In the form given the first term stands alone for the extreme negative range $(-\frac{1}{2}n<e<-\frac{1}{2}n+1)$, but when the centre is reached the law for positive errors is equally given by changing the sign of e and cutting off terms, or by leaving the sign of e unchanged and continuing the series term by term until $e=\frac{1}{2}n$. The formula preserves the symmetry automatically, and it seems a curious fact. It is explained by the vanishing of the expression when carried to $(n+1)$ terms in virtue of an identity well known in finite differences.

If the above frequency function in E be expressed in the artificial form

$$\frac{2 \cdot dE}{\pi(n-1)!}\int_0^\infty \left\{ E^{n-1} - n(E-1)^{n-1}\cos x + \binom{n}{2}(E-2)^{n-1}\cos 2x \right.$$

$$\left. - \binom{n}{3}(E-3)^{n-1}\cos 3x + \ldots \right\} \frac{\sin Ex}{x} dx,$$

the series being continued to $n+1$ terms, it represents the solution for *all* positive values of E without any further qualification.

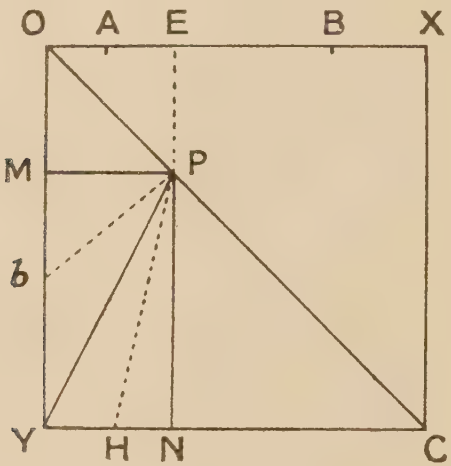
7. The problem bears an evident relation to the familiar question in probability concerning the total obtained in drawing a ticket n times from a bag containing p tickets marked 1, 2, . . ., p, when the ticket is put back between each drawing. But the deduction of the solution in this way would probably involve complicated questions of limits.

It is part of the fascination of questions in probability that the same question can generally be put in several different forms. Thus the question solved may be expressed in this way. Given n pieces of string of equal length, what is the probability of any given combined length when a piece is cut off each at random? This may be a familiar question * in the literature of the subject. Now Dr. Schlesinger has shown that when five pieces of string are taken, the distribution of the total lengths cut off is practically indistinguishable (among 1000 trials) from the probability function.

An obvious extension of the problem is made by taking pieces of string of different lengths, corresponding to items from separate tables with different degrees of accuracy. It would be out of place to pursue this question in detail. But one feature of it may be noticed. The graphical solutions apply when the square is replaced by a rectangle, the cube by a rectangular parallelepiped, and so on. The sections are still made by the line $x+y=e$, the plane $x+y+z=e$, . . . Thus the number of discontinuities is greatly increased, occurring as each corner of the figure is reached, whereas in the previous case sets of corners were reached simultaneously. The number of discontinuities in the frequency function is therefore increased to 2^n or $(1+1)^n$, while before it was equal to the number of terms in the expansion of $(1+1)^n$ or $n+1$. The consideration of such discontinuities may have this practical bearing, that perhaps there is too general a tendency to assume that the functions characteristic of natural processes must be continuous.

* The conjecture that the solution of some equivalent problem was known has soon been verified. The general form of solution was found by Laplace, who applied it to the observed inclinations in the planetary system. Cf. *Théorie Analytique des Probabilités*, p. 259. The same formula has also been applied to physical questions, such as the approach of atomic weights to integers.

8. Dr. Schlesinger's second problem may also be considered from a geometrical point of view. In this case a strict linear interpolation, for any fraction z, is made between two tabular numbers. If e_1, e_2 are the tabular errors, the error of the interpolated quantity is $ze_1 + (1-z)e_2$, where z has any value between 0 and 1. It is convenient to take 0 and 1 as the limits of the tabular error. On a line of unit length OX two points AB may be taken such that



$OA = e_1$, $OB = e_2$. If $OE = E = ze_1 + (1-z)e_2$, E is a random point between AB, since z is any proper fraction. The problem can therefore be stated in the form: any two points AB being taken on a unit straight line, and then another point E on the segment between them, the distribution of E along OX is required. Obviously it has a greater tendency towards the middle of OX than the direct tabular errors represented by A or B.

Let e_1 be the smaller of the two errors. It may be represented by the x co-ordinate of a point. The larger error may be represented by the y co-ordinate of the same point, and the whole of such points will cover evenly the triangle OYC, where $OY = 1$, since $y > x$. A third co-ordinate z is added, and (x, y, z) then represents any point in the prism of unit height on the base OYC. The locus of points corresponding to a given error E is given by

$$zx + (1-z)y = E,$$

a ruled surface of the second degree. The probability of the error being less than E is represented by the volume which this surface cuts from the prism towards the edge through O. In any section of the prism parallel to the base the section of the surface is a straight line. In the plane $z = 0$ it is $y = E$, or PM, where $OP = E\sqrt{2}$. In the plane $z = 1$ it is $x = E$, or PN. In intermediate

planes it is PG, PY, PH. Thus $OG = E/(1-z)$ when $1-z > E$, $YH = (E+z-1)/z$ when $E+z > 1$, and the section is PY when $z = 1-E$. Hence the volume cut off from the prism by the surface is

$$\int_0^{1-E} \tfrac{1}{2}E \cdot \frac{E}{1-z} \cdot dz + \tfrac{1}{2}E\{1-(1-E)\} + \int_{1-E}^{1} \tfrac{1}{2}(1-E) \cdot \frac{E+z-1}{z} \cdot dz$$

$$= -\tfrac{1}{2}E^2 \log E + \tfrac{1}{2}E^2 + \tfrac{1}{2}(1-E)E + \tfrac{1}{2}(1-E)^2 \log (1-E)$$

$$= \tfrac{1}{2}E - \tfrac{1}{2}E^2 \log E + \tfrac{1}{2}(1-E)^2 \log (1-E).$$

The whole volume of the prism is $\frac{1}{2}$. Hence the probability of an error between E and $E+dE$ is represented by the element of volume

$$-2dE\{E \log E + (1-E) \log (1-E)\},$$

or, on changing the range to $\pm\frac{1}{2}$ by writing $E - e = \frac{1}{2}$,

$$de\{(\tfrac{1}{2}+e) \log (\tfrac{1}{2}+e)^{-2} + (\tfrac{1}{2}-e) \log (\tfrac{1}{2}-e)^{-2}\},$$

which gives Dr. Schlesinger's result.

The probable error is given by that surface which cuts off one quarter of the volume of the prism. The equation

$$\tfrac{1}{4} = E - E^2 \log E + (1-E)^2 \log (1-E)$$

is satisfied by $E = 0{\cdot}313$. The probable error is therefore $e = \pm 0{\cdot}187$, as compared with $0{\cdot}25$ for a direct tabular quantity. The error of mean square has been seen to be $1/3\sqrt{2}$, or $0{\cdot}236$. The ratio of probable error to error of mean square is thus nearly $4/5$ as compared with $2/3$ for a Gaussian distribution. Nevertheless Dr. Schlesinger shows that for a sum of only two unrounded interpolations the Gaussian frequency is nearly reached.

As an example of the same solution applied to very different subject-matter the following may be mentioned. Two historical documents exist, undated except for the year. In the one a certain person is mentioned as being alive, in the other as being dead. He is equally likely to have died within a period of 137 days in the middle of the year as at any time in the remaining 228 days. The probability that he died on any particular day or in any particular month can be calculated by Dr. Schlesinger's formula. July is rather more than four times as likely as December, and it is about 36 to 1 against the death having happened in the latter month.

Dunsink Observatory:
1917 *November* 15.

The Short-period Variable RZ Cephei. By C. Martin and H. C. Plummer.

1. The variability of RZ Cephei was discovered by Miss Leavitt in 1907 from an examination of plates of the Harvard map of the sky.* Its discovery number was 131·1907, and other designations are B.D. +64°·1698, A.G.C. (Hels.-Gotha) 13339, and 223564 on the Harvard system.

2. The character of its period seems to have been first established by Zinner.† The elements given are

$$\text{J.D. } 2419742{\cdot}273 + 0{\cdot}30864\ E.$$

From the Dunsink observations a maximum is inferred at J.D. 2421496·616. These epochs can be reconciled by supposing 5684 periods to have elapsed, and the elements then become

$$\text{J.D. } 2421496{\cdot}616 + 0{\cdot}308646\ E,$$

which suit the Dunsink observations perfectly well. The period is thus about 7·4 hours, or among the shortest known.

3. Information relating to the comparison stars is given in Table I. The magnitudes m_1, m_2 are based on Schwarzschild's system for the χ Persei cluster, through two plates bearing duplicate exposures on this region and the region of RZ Cephei. The mean magnitudes m are adopted in the reduction of the variable, the plate residuals giving small mean corrections, which are incorporated in m'. From the same residuals, of which the extreme range is given in the last column, the probable error for a single plate is inferred and shown in the preceding column.

Table I.

B.D.	Pl. 1800. m_1.	Pl. 1807. m_2.	Mean. m.	Corr.	Final. m'.	P.E.	Range.
+63°·1868	8·65	8·57	8·61	−·006	8·60	±·016	·12
+64·1699	9·32	9·27	9·30	−·004	9·30	±·022	·15
+63·1873	9·63	9·60	9·61	+·007	9·62	±·017	·14
+64·1703	9·97	9·90	9·94	+·001	9·94	±·011	·07
+64·1700	10·20	10·15	10·17	−·009	10·16	±·016	·11

4. The results from 50 exposures on the variable are given in Table II. In the second and third columns are recorded the astronomical date and G.M.T., together with the Julian day and decimal fraction, corresponding to the middle of each exposure, the length of which is shown in the fifth column. The phase in days, given in the fourth column, is based on the elements derived in

* *H.C.O. Circular*, 130; *A.N.*, 4196.

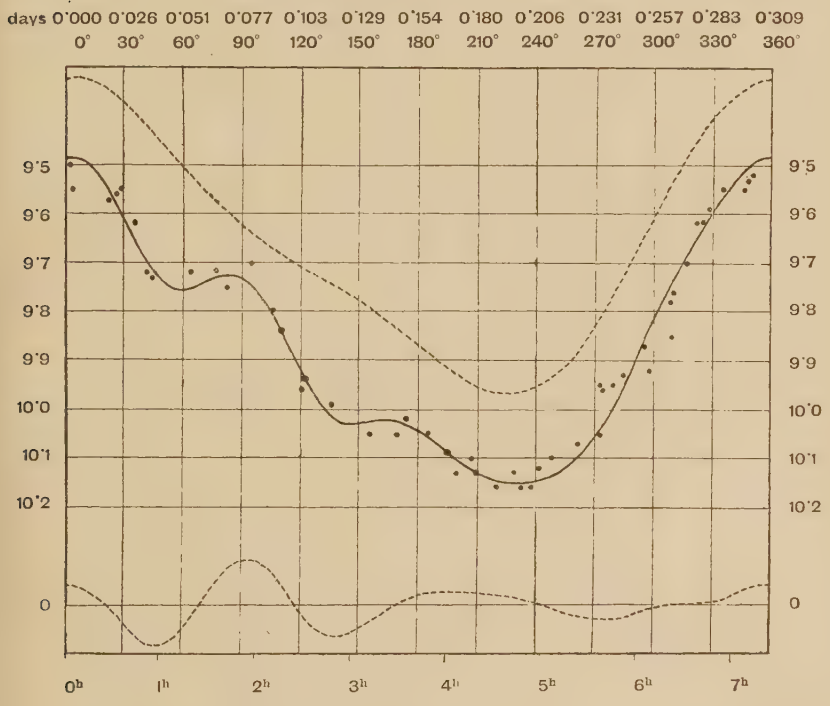
† *V.J.S.*, li. p. 262.

§ 2. The curve from which the differences shown in the seventh column are derived has the equation

$$m = 9{\cdot}86 + 0{\cdot}29 \sin(\theta + 250^\circ) + 0{\cdot}08 \sin(2\theta + 302^\circ) + 0{\cdot}04 \sin(3\theta + 343^\circ) + 0{\cdot}03 \sin(4\theta + 257^\circ) + 0{\cdot}02 \sin(5\theta + 177^\circ),$$

where θ has the period 2π corresponding to the period of the star and is reckoned from the assumed epoch of maximum. According to these residuals the probable error of a measured magnitude of the variable is $\pm{\cdot}022$.

5. The same curve with the observations is represented in fig. 1. This figure contains also two other curves. The upper, raised by $0^m{\cdot}2$, exhibits the result of omitting the third, fourth,



Photographic Light Curve of RZ Cephei.

and fifth harmonics. It seems interesting because it shows the very steady fall on the descending branch, perhaps as near an approach to a linear gradient as two harmonics are capable of reproducing. The bottom curve of the same figure represents the sum of the three later harmonics separately. It suggests clearly a wave with a period equal to one quarter of the whole period of the star, active in the first half of the period and strongly damped in the second.

TABLE II.

Serial No.	Date G.M.T.	Julian Day.	Phase in Days.	Exposure.	Magn.	Obs. —Curve.	Images.
	1917 d h m	2421		min.			
1	Sept. 11 11 43	483·489	·145	40	10·05	+·03	Good.
2	20 11 10	492·465	·170	40	10·13	+·03	Good.
3	24 11 40	496·486	·179	40	10·13	·00	Fair.
4	24 12 21	496·514	·207	40	10·12	−·03	Fair.
5	24 13 12	496·550	·243	40	9·93	−·03	Good.
6	24 14 0	496·583	·276	40	9·62	−·03	Good.
7	24 14 50	496·618	·002	40	9·50	+·01	Good.
8	Oct. 9 9 48	511·409	·287	13	9·55	−·02	Fair.
9	10 11 5	512·462	·105	40	9·94	·00	Good.
10	10 11 46	512·490	·133	40	10·05	+·02	Good.
11	10 12 34	512·524	·167	40	10·09	·00	Good.
12	10 13 16	512·553	·196	40	10·13	−·02	Good.
13	12 9 54	514·413	·204	40	10·16	+·01	Good.
14	12 10 35	514·442	·233	40	9·95	+·01	Good.
15	12 11 23	514·474	·265	40	9·85	+·11	Good.
16	12 12 13	514·509	·300	40	9·52	+·02	Good.
17	12 13 0	514·542	·025	40	9·55	−·05	Good.
18	13 11 33	515·481	·038	40	9·73	+·01	Good.
19	13 12 20	515·514	·071	40	9·75	+·02	Good.
20	13 13 8	515·547	·104	40	9·96	+·04	Good.
21	17 9 10	519·382	·235	40	9·96	−·07	Good.
22	17 9 51	519·411	·264	40	9·78	+·04	Fair.
23	17 10 40	519·445	·298	40	9·53	+·03	Fair.
24	17 11 28	519·478	·022	40	9·56	−·02	Good.
25	17 12 16	519·511	·055	40	9·72	−·03	Good.
26	17 13 6	519·546	·090	40	9·80	−·01	Good.
27	18 10 2	520·418	·036	40	9·72	+·02	Good.
28	18 10 43	520·448	·066	40	9·72	−·01	Good.
29	23 9 56	525·414	·094	22	9·84	·00	Fair.
30	Nov. 7 8 2	540·334	·199	40	10·16	+·01	Good.
31	7 8 48	540·367	·232	40	10·05	+·01	Good.
32	7 9 37	540·401	·266	40	9·76	+·03	Good.
33	10 8 54	543·371	·149	40	10·02	−·01	Good.
34	10 9 36	543·400	·178	40	10·10	−·03	Good.
35	10 10 24	543·434	·212	40	10·10	−·04	Good.
36	10 11 4	543·461	·239	40	9·95	−·05	Good.
37	10 11 50	543·493	·271	36	9·70	+·01	Good.
38	10 12 27	543·519	·297	36	9·55	+·04	Good.

TABLE II.—*continued.*

Serial No.	Date G.M.T.	Julian Day.	Phase in Days.	Exposure.	Magn.	Obs. —Curve.	Images.
	1917 d h m	2421		min.			
39	Nov. 10 13 10	543·549	·019	36	9·57	+·02	Good.
40	12 7 51	545·328	·255	36	9·92	+·09	Good.
41	12 8 25	545·351	·278	30	9·62	−·01	Good.
42	14 7 33	547·315	·081	40	9·70	−·05	Good.
43	14 8 25	547·351	·117	40	9·99	−·02	Good.
44	14 9 25	547·393	·159	40	10·05	−·01	Good.
45	14 10 9	547·423	·189	40	10·16	+·01	Good.
46	14 11 0	547·458	·224	40	10·07	−·03	Good.
47	14 11 40	547·486	·252	40	9·87	+·01	Good.
48	14 12 23	547·516	·282	30	9·59	−·01	Fair.
49	14 13 5	547·546	·003	40	9·55	+·06	Fair.
50	14 13 45	547·574	·031	40	9·62	−·04	Fair.

6. Here a point requires notice. The period of the star is only 7·4 hours, and the majority of the exposures are 40 minutes in length. This is a necessary circumstance of the method, especially as the temperature at the time of observation was low, and the plates were therefore less sensitive than in warmer conditions. The fact that the exposures thus covered one-eleventh of the whole period might be thought fatal to accurate results, and indeed some hesitation was felt in placing the star on the programme. The effect may be gauged to some extent thus. Let the true magnitude of the star be represented by

$$m = \sum_r a_r \sin(r\theta + \gamma_r).$$

Let $\theta \pm \tau$ represent the beginning and end of an exposure. Then the measured magnitude will be approximately

$$\frac{1}{2\tau}\int_{\theta-\tau}^{\theta+\tau} m\,d\theta = -\left[\sum \frac{a_r}{2r\tau}\cos(r\theta+\gamma_r)\right]_{\theta-\tau}^{\theta+\tau}$$

$$= \sum a_r \sin(r\theta+\gamma_r)\cdot\frac{\sin r\tau}{r\tau}.$$

The coefficients of the successive harmonics are therefore diminished by a factor which vanishes when $r = \pi/\tau$, but up to this point the phases of the harmonics are undisturbed. In the present case $\pi/\tau = 11$, and the harmonics in the neighbourhood of this order should very nearly vanish, as in fact they practically do. From their absence nothing can be inferred, and the result is a rather smoother curve than we have obtained for some other stars. This may be the artificial effect of the long exposures.

But the earlier harmonics should be little affected. The factors of reduction for the first six harmonics are:

(1) ·989, (2) ·946, (3) ·882, (4) ·796, (5) ·693, (6) ·578,

and division by these factors should restore approximately the true coefficients. The formula in § 4 would not be very seriously altered.

7. The reason for giving separate attention to the higher harmonics is that they apparently form a connected group. Their effect is visible in the transient wave on the descending branch of the curve. But the connection is marked in another way. Their phases, which are not affected by the length of the exposures, have a common difference of about $-83°$. This can be traced further, for the phase of the sixth harmonic is 96°. The term has been omitted because its coefficient is under 0·01, but it would be nearer 0·02 were it not nearly halved by the long exposures. Thus the phases of four terms are practically in arithmetical progression, and it seems natural to class them together. The question now presents itself whether this group has, or has not, any relation with the first two harmonics. It may be decided by the phase of the third harmonic. But a preliminary change of notation will be convenient. Our results have been expressed in the form

$$m = \Sigma a_r \sin (r\theta + \gamma_r),$$

where θ is reckoned from the approximate maximum. By a change of epoch the form may be altered to

$$m = \Sigma a_r \cos (r\theta - \phi_r),$$

where

$$\phi_r = \tfrac{1}{2}(r+1)\pi + r\gamma_1 - \gamma_r, \quad \phi_1 = \pi.$$

The phase of the first harmonic is now fixed.

8. In Table III. are collected those of our results which bear on this question. The first thing to be noticed is the very small range in ϕ_2. It is no more than 25° among the eight stars with periods ranging from 7 hours to 14 days. The range in ϕ_3 is greater, but it is under 50° among the five members of what may conveniently be called the merihemeral group, which embraces the cluster type. The range is extended when stars of the δ Cephei type outside the merihemeral class are included, but even then the value of ϕ_3 seems to be by no means unrestricted. The mean value of ϕ_3 in the narrower class is 52°, while the value for RZ Cephei is 47°. Thus it would appear that the train of secondary waves in the light variation of this star is very distinctly connected with the simple primary wave, and that the connection depends on some feature which is more or less common to the group.

TABLE III.

Star.		γ_1.	γ_2.	γ_3.	ϕ_1.	ϕ_2.	ϕ_3.	No.
	h	°	°	°	°	°	°	
RZ Cephei	(7·4)	250	302	343	180	108	47	1
XZ Cygni	(11·2)	231	239	258	,,	133	75	2
RR Lyræ	(13·6)	234	260	286	,,	118	56	3
SW Draconis	(13·7)	231	242	307	,,	130	26	4
SU Draconis	(15·9)	240	272	302	,,	118	58	5
	d							
V Lacertæ	(5·0)	229	260	294	,,	108	33	6
X Lacertæ	(5·4)	218	235	173	,,	111	121	7
RW Cassiopeiæ	(14·8)	233	260	234	,,	116	105	8

9. It is now generally accepted that what we have called the merihemeral group is only a branch of the δ Cephei type. The range of periods shown by the eight stars of Table III. is from 7 hours to 14 days, and this is capable of extension at both ends. As it stands, it gives a ratio of nearly 50 to 1, and if the time scale be extended in the same ratio it will cover two years and embrace the long-period variables. In any case it is of interest to compare the results shown in Table III. with those obtained by Mr. T. E. R. Phillips and described in his presidential address (1916) to the British Astronomical Association. When this is done the fact is at once seen that, according to the evidence of the phases, these stars belong quite distinctly to the division which he has distinguished as Group II. Perhaps the resemblance will be brought out by matching our stars among the long-period variables. The selected material for comparison is shown in Table IV., where the numbers in the last column refer to those in the last column of Table III.

TABLE IV.

Star.	ϕ_1.	ϕ_2.	ϕ_3.	Compare:
The Sun (11 years) . . .	180	106°	36°	1, 6
o Ceti (332 days)	,,	118	59	3, 5
R Comæ Ber. (362) . . .	,,	122	33	4
U Orionis (368)	,,	126	106	7, 8
R Aquilæ (320)	,,	134	89	2

This represents only a small section of one of the two linear branches into which the long-period variables divide. But it is difficult to deny all significance to the resemblances, which indeed are visibly reflected in the shape of the light curves. The phases cannot, however, be deduced from the curves by mere inspection.

10. The results of the comparison of Table IV. with Table III. are at least very curious. The closest resemblance of all from this point of view is between Mira on the one hand and RR Lyræ and SU Draconis on the other. Plotted on a (ϕ_2, ϕ_3) diagram the greatest distance between the points compared is 21° in the case of U Orionis and X Lacertæ, and this would become quite small if the phases of U Orionis were based on Harvard observations alone. The introduction of the sun-spot curve is interesting on more than

one account, especially as it is in no sense dragged in, but used for the simple reason that its representative point is recorded in Mr. Phillips's (ϕ_2, ϕ_3) diagram. Now curves of simple type may well have features in common without implying any community of origin, and one might hesitate to see anything more than an accidental resemblance between classes of stars differing so widely in period, range of magnitude, and spectral type as RR Lyræ and o Ceti. But if there is any reason to connect the Sun, possessing a minute range, an exceedingly long period, and belonging to type G, with the ordinary long-period variables, there is so much the less reason to interrupt the sequence of variability in the opposite direction of the scale of periods. The inclusion of the Sun, in fact, makes it unnecessary to regard period, range, or spectral type as delimiting factors.

11. We are thus led to contemplate the possibility of something like a common process in operation in stars differing so widely as RZ Cephei, with its period of $7^h.4$, or indeed XX Cygni with a period of $3\frac{1}{4}^h$, and stars with a range of some 10 magnitudes and periods of the order of a year, or even the Sun with its period of 11 years. There are certainly features which favour this view. Some stars are more irregular in period and range than others, but probably all are irregular to some extent, while some are so irregular that all trace of period is lost. Those changes of spectral type which Mr. Shapley has shown to accompany short-period light changes are shared by the Sun, for the spectrum of a sun-spot approximates to that of an M type star. Will it be objected that the short-period variable is a spectroscopic binary, while the long-period variable is not? We have already dissented from the binary interpretation of the spectroscopic evidence, and discussed the star RR Lyræ on the basis of an alternative hypothesis. Mr Shapley has also advanced arguments against the binary interpretation. The absence of a similar appearance of variable radial velocity in the long-period variable would be sufficiently explained by the slower development of the effective process. But the effect is probably not altogether absent. The variability of α Orionis, irregular as it is, led one of us ten years ago to look definitely for evidence of variable radial velocity. The suspicion was immediately substantiated, and since that time efforts have been made to treat the star as a binary and to determine the orbit. The small success attending these efforts is entirely in accordance with the writer's expectation, for the displacements in the spectrum are not to be attributed to orbital motion. They are directly connected with the mechanism which produces the light changes, and these must be regarded as irregular. When the changes are rapid and approximately periodic, the displacements in the spectrum simulate the effect of orbital motion. But the possibility of so regarding them is no more an argument that the view is true than the evidence of those many cases where the velocity curve cannot be explained by elliptic motion and it has been necessary to admit the existence of periodic disturbances.

12. On the whole, there seems to be no very cogent reason against the view that, outside the eclipsing systems, the great majority of variable stars manifest the operation of one essentially uniform process in nature. That is the outcome of the struggle towards adjustment between radiative expenditure and mechanical equilibrium. So much indeed has long been evident. But the suggestion contained in Mr. Phillips's address takes us a step further by showing how the one process works out with a certain regularity of detail under varying circumstances. It now appears that something of the same regularity can be traced on a far wider scale. It may well be convenient to preserve for some purposes a classification which depends on the length of time covered by the changes, but it seems possible that the distinctions are really artificial, and that the great majority of variable stars may be embraced in one broad scheme irrespective of the length of period. If this proves to be so, the whole subject will gain both in unity and interest.

Dunsink Observatory:
1917 *December* 3.

Variations in the Fourteen Months' Component of the Polar Motion. By Hisashi Kimura, Assoc.R.A.S.

I completed this work two years ago, but it was laid aside until the appearance of the recent paper of the late Dr. Backlund on the same problem (*Monthly Notices*, lxxvii. p. 2) suggested the advisability of publishing my results.

The materials available for a determination of the elements (radius and period) of the fourteen months' term in polar motion consisted of the two series of observations made at Greenwich * and Pulkowa † from the epoch 1825 till 1890, and, after this, those of the latitude-variation investigations of Messrs. Albrecht and Wanach.

The process of finding the fourteen months' term was that of successive approximation, as commonly employed, by the alternate elimination of fourteen months' and annual terms, taking means of six or seven years (rarely five or eight years), acording to the lengths of recurrence of maximum or minimum of the latitude-variation. To the maximum epochs of the fourteen months' motion for the Pulkowa observations reduction to the meridian of Greenwich was applied on account of the difference of longitudes. The combination of two results from both observatories

* "Pond's Double Altitude Obs. 1825–35," *A.J.*, 315; "Greenwich Mural-Circle Obs. 1836–51," *A.J.*, 320; "Greenwich Transit-Circle Obs. of Four Close Circumpolars, 1851·0–91·6." By W. G. Thackeray (*M.N.*, liii.).

† "Recherches definitives sur les variations de la latitude de Poulkovo d'après les observations anciennes faites au grand cercle vertical," par A. Ivanof (*Bull. de l'Académie Impériale des Sciences de St.-Pétersbourg*, 1895, No. 4); "Variations de la latitude de Poulkovo observées au grand cercle vertical dans les années 1882–91," par M. Nyrén (*Bull. de l'Académie Impériale des Sciences de St.-Pétersbourg*, tome vii.).

during common epochs was made by taking the mean. Here it ought to be noted that, although the data of ancient observations contain several considerable systematic errors, most of them being of annual period, the fourteen months' motion thus found is much more likely to be free from such kind of errors.

Thus the following data for the radius (R) and the period (P) of the fourteen months' term were obtained :—

TABLE I.

Maximum Epoch referred to the Greenwich Meridian.	Radius.	Interval between Two Maxima.	Number of Revolutions.	Mean Date.	Period.
	"	y			y
1828·36	0·119	4·77	4	1830·75	1·193
1833·13	0·106	6·92	6	1836·59	1·153
1840·05	0·040	5·59	5	1842·85	1·118
1845·64	0·097	9·27	8	1850·28	1·159
1854·91	0·197	6·15	5	1857·99	1·230
1861·06	0·156	5·98	5	1864·05	1·196
1867·04	0·218	5·92	5	1870·00	1·184
1872·96	0·185	4·68	4	1875·30	1·170
1877·64	0·209	8·20	7	1881·74	1·171
1885·84	0·178	5·90	5	1888·79	1·180
1891·74	0·183	5·98	5	1894·73	1·196
1897·72	0·119	5·94	5	1900·69	1·188
1903·66	0·134	4·70	4	1906·01	1·175
1908·36	0·215	3·61	3	1910·17	1·203
1911·97	0·233				

In the above table the period about 1840 is uncertain on account of the smallness of amplitude at that time, the amount being comparable with that of accidental errors.

The values given in Table I. were at first plotted, and smoothed curves were then drawn through them. The ordinary method of periodic analysis was afterwards carried out, based on seventeen ordinates read off at equal intervals of five years over the whole period. The lengths of several periods indicated by the curves were found according to the methods given by Schuster and Kühnen.

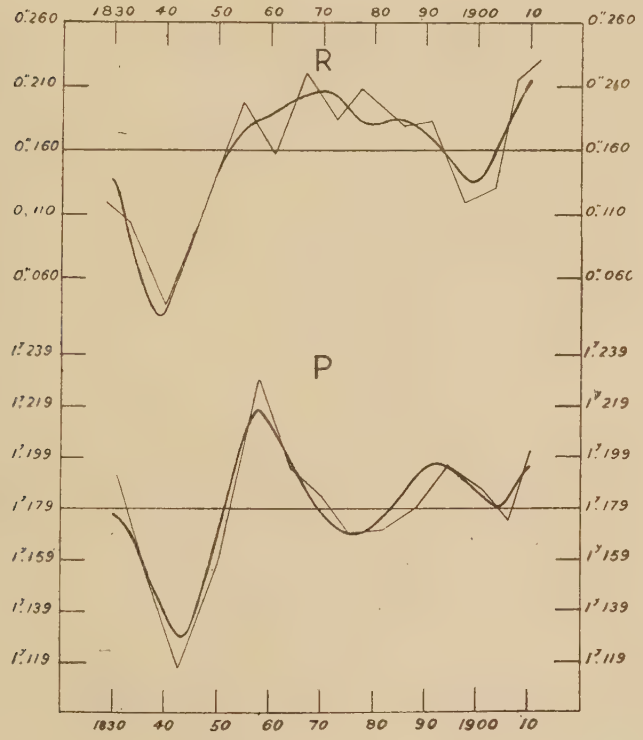
The process gave the following formulæ for R and P :—

$$\begin{aligned} R = {} & 0''\cdot160 + 0''\cdot049 \cos\left\{\frac{2\pi}{80}(t-t_0) - 57^\circ\right\} \\ & + 0''\cdot044 \cos\left\{\frac{2\pi}{50}(t-t_0) - 302^\circ\right\} \\ & + 0''\cdot027 \cos\left\{\frac{2\pi}{26\cdot67}(t-t_0) - 168^\circ\right\} \\ & + 0''\cdot021 \cos\left\{\frac{2\pi}{22}(t-t_0) - 352^\circ\right\} \end{aligned}$$

$$P = 1^{y}\cdot179 + 0^{y}\cdot018 \cos\left\{\frac{2\pi}{80}(t-t_0) - 66^\circ\right\}$$
$$+ 0^{y}\cdot023 \cos\left\{\frac{2\pi}{50}(t-t_0) - 274^\circ\right\}$$
$$+ 0^{y}\cdot021 \cos\left\{\frac{2\pi}{26\cdot67}(t-t_0) - 232^\circ\right\}$$
$$+ 0^{y}\cdot011 \cos\left\{\frac{2\pi}{22}(t-t_0) - 81^\circ\right\}$$

where $t_0 = 1870\cdot0$ and t is in units of a year.

The following table gives a comparison between the observed and the calculated values for R and P. These are sufficiently



in agreement, as indicated in the last column, and also in the accompanying diagram. Still closer approaches could be obtained by introducing some terms of shorter periods.

TABLE II.

Radius (*in units* 0″·001).

0″·160 + tabular values.

Date.	$^{y}_{80.}$	$^{y}_{50.}$	$^{y}_{26{\cdot}67.}$	$^{y}_{22.}$	Sum of the Four Columns. Cal.	Sum of the Four Columns. Obs. (smoothed).	Obs.-Cal.
1830·0	−27	−28	+25	+ 6	− 24	− 47	−23
1835·0	−40	−42	+ 1	−19	−100	− 71	+29
1840·0	−48	−41	−24	−12	−125	−120	+ 5
1845·0	−48	−23	−20	+16	− 75	− 71	+ 4
1850·0	−41	+ 3	+ 8	+16	− 14	− 19	− 5
1855·0	−28	+28	+27	−12	+ 15	+ 36	+21
1860·0	−10	+42	+14	−20	+ 26	− 5	−31
1865·0	+ 9	+41	−16	+ 6	+ 40	+ 38	− 2
1870·0	+27	+23	−26	+21	+ 45	+ 39	− 6
1875·0	+40	− 3	− 6	0	+ 31	+ 35	+ 4
1880·0	+48	−28	+22	−21	+ 21	+ 39	+18
1885·0	+48	−42	+23	− 6	+ 23	+ 20	− 3
1890·0	+41	−41	− 3	+19	+ 16	+ 21	+ 5
1895·0	+28	−23	−26	+12	− 9	− 13	− 4
1900·0	+10	+ 3	−18	−16	− 21	− 36	−15
1905·0	− 9	+28	+11	−16	+ 14	− 3	−17
1910·0	−27	+42	+27	+12	+ 54	+ 62	+ 8

Period (*in units* $0^{y}{\cdot}001$).

$1^{y}{\cdot}179$ + tabular values.

1830·0	− 7	−22	+16	+10	− 3	+ 11	+14
1835·0	−13	−23	+19	+ 4	− 13	− 17	− 4
1840·0	−17	−15	0	− 9	− 41	− 47	− 6
1845·0	−18	− 2	−19	− 7	− 46	− 49	− 3
1850·0	−17	+13	−16	+ 7	− 13	− 22	− 9
1855·0	−12	+22	+ 7	+ 9	+ 26	+ 22	− 4
1860·0	− 7	+23	+21	− 4	+ 33	+ 40	+ 7
1865·0	0	+15	+10	−10	+ 15	+ 15	0
1870·0	+ 7	+ 2	−13	+ 2	− 2	+ 5	+ 7
1875·0	+13	−13	−20	+11	− 9	− 8	+ 1
1880·0	+17	−22	− 4	+ 1	− 8	− 8	0
1885·0	+18	−23	+18	−10	+ 3	− 1	− 4
1890·0	+17	−15	+18	− 4	+ 16	+ 3	−13
1895·0	+12	− 2	− 3	+ 9	+ 16	+ 16	0
1900·0	+ 7	+13	−20	+ 7	+ 7	+ 10	+ 3
1905·0	0	+22	−13	− 7	+ 2	0	− 2
1910·0	− 7	+23	+ 9	− 9	+ 16	+ 23	+ 7

Conclusions.

1. It is a remarkable fact that the periods found among the expressions for R and P are nearly odd submultiples of a main period of about 240 years, as shown below:—

y.			y.
80	×	3	= 240
50*	×	5	= 250
27	×	9	= 243
22	×	11	= 242

2. The reappearance of the main period of the same length found in the case of sun-spot relative numbers † explains some close relations between the two phenomena. The only notable discordance is the absence of several periods of eleven years, and near it, in the fourteen months' motion, which are especially considerable in sun-spots. This fact strengthens greatly my opinion quoted previously in my paper on "Sun-spot Relative Numbers."

3. The predicted values for the coming years suggest that a pretty considerable minimum radius will occur about 1930.

4. The present results of analysis are of course only provisional. I did not, however, endeavour to make any further approximation, because the raw materials which form the basis of the investigation are full of unavoidable errors, and it is unlikely that further investigation would lead to any sensible improvement.

Mizusawa International Latitude Observatory:
1917 *October* 11.

Observations of α Orionis. By T. W. Backhouse.

In the *Monthly Notices of the Royal Astronomical Society*, vol. lix. pp. 194–199, I gave details of my observations of the brightness of α Orionis from 1895–1898. The following is an abridgment of a paper containing observations supplementary to them, the complete manuscript being filed with the Royal Astronomical Society.

The observations were made at Sunderland with the naked eye, except where otherwise stated.

The comparison stars are given in the following list, with the designations here adopted for the sake of brevity, the letters as

* 49y is probable value, but for the convenience of calculation 50y was taken.

† *Monthly Notices*, lxxiii. p. 544.

far as they go being the same as in the paper on Baxendell's observations of this star (*Monthly Notices*, vol. lxxi. p. 702).

Star.	Designation.	Magnitude.
β Orionis	B	0·28
α Canis Minoris	C	0·47
α Tauri	D	1·23
β Geminorum	E	1·35
α Geminorum	F	1·61
β Tauri	G	1·66
γ Orionis	H	1·80
α Orionis	X	...

The magnitudes are taken from a Revision of the Harvard Photometry (*H.A.*, vol. xliv. part i.), except α Tauri, β Geminorum, and γ Orionis, which are taken from my observations in conjunction with and based on the Harvard magnitudes for white stars.

The method of comparison is that of Argelander, namely, by "step estimations." The numbers expressing the steps have been treated as differences of magnitude on a scale indicated in each case by the observation itself. In most cases, when the variable is placed between two given stars, the magnitude is obtained by simple interpolation between the two assumed magnitudes proportional to the intervals denoted by the steps recorded. When α Orionis is compared with one star only, the step value is assumed, and is taken as equal to 0·1 magnitude, unless there are other observations near the same time from which the value can be calculated.

Allowance has been made for atmospheric absorption due to the altitude of the stars and the sky conditions.

In the majority of the observations the variable has been compared with other stars than those here mentioned, though in most cases the magnitude depends only on the stars given.

The mean magnitude during the period covered by these observations was 1·01, ranging from 0·49 to 1·64. The difference of this from the mean 0·67 (*M.N.*, vol. lxxi. p. 704) no doubt arises mainly from personal equation, owing to colour.

There are some cases where observations differing little in time show considerable difference in the magnitude of α Orionis. This is not likely to represent a true change in brightness, but is probably due to error in observation—it may be to the existence of unsuspected cloud, or to irregularity in the transparency of the atmosphere. Among such cases are those of two observations of 1916 January 2, which differ by ·21 mag.; and on October 31, where they differ by ·55 mag. In the latter case a small part of the discrepancy may be accounted for by an overestimation of the amount of mist.

Date.	Time.		Mag.
1887	h m		
III. 23	7 30	C 4 X	0·85
,,	9 23	C 4·5 X	0·78
27	8 6	C 5·5 X 5·5 D	0·85
,,	8 47	C 5 X 5 D	0·83
XII. 21	10 28	D 3 X	1·40
,,	11 44	D 3 X 3 H	1·49
1888			
I. 2	12 55	D 4·5 X 4·5 H	1·54
9	11 10	D 3·5 X 2 H	1·59
19	11 21	D 2 X 3·5 H	1·45
II. 10	10 10	C 4 X 2 D	0·97
11	10 53	C 5 X 2 D	1·01
14	10 32	C 5 X 2 D	1·01
IV. 4	8 27	C 7 X 6 D	0·86
X. 16	16 55	C 1·5 X 3 D	0·73
XI. 13	17 12	C 5·5 X 4 D	0·89
XII. 10	14 5	C 3 X 4 D	0·78
25	14 7	C 7 X 3·5 D	0·98
1889			
I. 2	11 33	C 4 X 2·5 D	0·94
20	7 52	C 2 X 1 D	0·99
21	8 58	D 1·5 X 5 G	1·25
22	10 54	C 7 X 1 D	1·12
III. 26	8 43	C 5·5 X 3 D	0·96 *a*
XI. 24	17 20	C 3·5 X 3 D	0·86
XII. 18	13 40	C 2 X 4·5 D	0·69
1890			
III. 8	8 0	C 2 X 5 D	0·68
12	10 18	C 3 X 5 D	0·72
XI. 7	14 45	C 1·5 X 3·5 D	0·72
14	16 10	C 2 X 5·5 D	0·65
XII. 18	12 35	C 3 X 4·5 D	0·96
,,	13 10	C 3 X 4·5 D	0·77
1891			
III. 2	9 12	C 3·5 X 4·5 D	0·79
3	7 30	C 3·5 X 4·5 D	0·81
,,	9 33	C 4 X 4 D	0·84
X. 1	14 56	C=X 4 D	0·73
XI. 29	16 43	C 3 X 5 D	0·71
,,	17 28	C 3·5 X 3·5 D	0·82
XII. 7	11 33	C 3·5 X 3·5 D	0·90 *b*
29	12 0	C 2·5 X 3 D	0·82
1892			
I. 22	12 49	C 4·5 X 3 D	0·91
29	11 43	C 4 X 4 D	0·82
,,	12 3	C 7 X 4 D	0·94
IX. 27	15 50	C 3·5 X 1·5 D	0·99
X. 25	14 20	C 4 X 3 D	0·92
XI. 16	11 59	C 4 X 2 D	0·98
,,	17 2	C 7 X 2 D	1·09
26	17 28	C 11 X 2 D	0·98 *c*
29	14 50	D 2·5 F=X 0·5 G	1·46
,,	15 54	D 0·5 F 0·5 X 1 G	1·42
XII. 19	10 12	C 4 X 0·5 D	1·11 *d*
1893			
I. 13	10 2	X=D	1·20 *e*
20	11 2	C 7·5 X 1·5 D	1·12 *e*
II. 5	9 17	C 5 X 1 D	1·10 *f*
9	8 43	X=D	1·23 *g*
III. 9	9 20	C 6 X 2 D	1·05
16	8 31	C 8 X 2·5 D	1·05
X. 22	16 20	C 1 X 6 D	0·60
XII. 9	11 40	C 3 X 4 D	0·84 *b*
11	11 50	C 4·5 X 5 D	0·86 *b*
1894			
I. 28	11 30	C 6 X 3·5 D	0·95
II. 3	11 8	C 5·5 X 4 D	0·89
7	9 52	C 6 X 3·5 D	0·94
24	11 5	C 7 X 4 D	0·96
26	10 28	C 6 X 5·5 D	0·85
27	9 8	C 5·5 X 5 D	0·86
III. 1	8 20	C 7 X 6 D	0·88
8	7 35	C 6 X 4 D	0·92
,,	8 25	C 6 X 4·5 D	0·90

a At Bainbridge, N. Yorkshire, alt. 700 ft.
b At Torquay. *c* Very uncertain. *d* Off Galicia.
e At Puerto Orotava, Teneriffe, alt. 300 ft.
f At Tangier. *g* At Granada, alt. 2600 ft.

Date.	Time. h m		Mag.
1894			
III. 25	8 39	C 4·5 X 6 D	0·77
X. 28	14 15	C 1 X 4 D	0·69 *a*
XI. 3	10 12	X=D	0·65
,,	16 21	C 2 X 6 D	0·65
8	9 20	X=D	0·49
,,	9 31	X=D	0·53
9	16 49	C 2 X 6 D	0·62
29	12 30	C 2·5 X 3·5 D	0·83
,,	16 32	C 6 X 4 D	0·92
XII. 7	16 3	C 5 X 5 D	0·83
26	13 2	C 4 X 4 D	0·85
1897			
XII. 17	9 25	C 2 X 4 D	0·77 *b*
25	8 15	C 1·5 X 4·5 D	0·69 *c*
1898			
I. 17	3 10	X=C	0·55 *d*
II. 12	4 17	C 3·5 X 4 D	0·84 *e*
III. 12	6 30	C 4·5 X 3 D	0·95 *c*
XII. 27	11 33	C 4·5 X=D	1·19
1899			
I. 4	12 30	C 6 X 1·5 D	1·08
11	11 28	C 5 X 1·5 D	1·04
13	12 40	C 7 X 1·5 D	1·10
16	13 10	C 8 X 1 D	1·18
II. 12	11 22	C 7 X 1 D	1·17
III. 29	9 42	C 5 X 3·5 D	0·91
XI. 8	16 28	C 4 X 3 D	0·90
XII. 1	11 33	C 4·5 X 4 D	0·90
1900			
I. 27	7 36	C 2 X 4 D	0·82 *f*
XII. 21	13 18	C 3 X 4 D	0·80
1901			
I. 22	12 50	C 4 X 4 D	0·83
XI. 14	12 40	C 3 X 4 D	0·88
,,	15 17	C 4 X 4·5 D	0·82
XI. 16	18 0	C 6 X 4 D	0·93

Date.	Time. h m		Mag.
1902			
I. 2	11 15	C 2 X 5 D	0·71
II. 11	10 50	C 5 X 5 D	0·84
12	11 20	C 6 X 5 D	0·86
IV. 2	8 40	C 6 X 5 D	0·86
,,	10 4	C 8 X 6 D	0·85
XII. 6	14 10	X 5 D	0·72
28	11 30	C 1 X 6 D	0·63 *a*
29	13 15	C 1·5 X 6 D	0·61
31	13 30	C 2·5 X 8 D	0·62
1903			
I. 2	14 7	C 4 X 6·5 D	0·73
7	14 45	C 5 X 9 D	0·68
28	11 25	C=X 8 D	0·44
II. 15	9 0	X 0·5 C 7·5 D	0·46 *g*
17	10 22	C 0·5 X 7 D	0·52
21	9 15	C 0·5 X	0·51
23	9 25	C=X 7·5 D	0·45
27	10 30	C 1 X 9 D	0·45
III. 27	8 30	C 2·5 X 9 D	0·57
28	8 20	C 2 X 8 D	0·57 *h*
XI. 14	15 30	C 4 X 3 D	0·90
15	15 30	C 3 X 5 D	0·75
18	16 40	C 6·5 X 4·5 D	0·92
21	11 37	X 3 D	0·82 *i*
29	17 20	C 6 X 4 D	0·92
1904			
I. 10	10 15	C 2 X 4 D	0·77
,,	12 8	C 3·5 X 5 D	0·78
11	13 47	C 6 X 5 D	0·87
II. 18	10 26	C 4 X 4·5 D	0·81 *j*
IV. 3	9 27	C 5 X 5 D	0·83
XI. 7	11 45	D 3·5 X	1·42
,,	16 15	D 2·5 E 0·5 X 2 F 1 G	1·34
14	13 10	D 5 X 0·5 E	1·30
XII. 5	11 55	D 3 X=E	1·29
6	16 7	D 1 G 1 X	1·58

a At Aysgarth, N. Yorkshire, alt. 670 ft.
b Between Straits of Bonifacio and Messina.
c In Red Sea.
d At Bhopal, India, alt. 1720 ft.
e At Darjeeling, India, alt. 6945 ft.
f At Bishop Auckland.
g At West Hartlepool.
h At Hawes, N. Yorkshire, alt. 700 ft.
i At Redcar, Yorks.
j At Torquay.

Date.	Time. h m		Mag.
1905			
I. 6	12 5	X 0·5 D	1·18
9	12 50	C 8 X 0·5 D	1·20
11	12 8	C 8 X 1·5 D	1·11
26	11 37	C 9 X 1 D	1·16
30	10 45	C 7 X 1·5 D	1·09
,,	12 0	C 9 X 2 D	1·11
II. 10	12 10	C 9 X 2 D	1·13
III. 9	10 14	C 9 X 2·5 D	1·09
VIII. 31	14 44	D 7 H=X 1 F=E	1·37 *a*
,,	15 4	D 2·5 X	1·24 *a*
IX. 7	17 25	X=D	1·18 *b*
14	16 25	D 0·5 X	1·18
X. 26	13 4	C 4 X=D	1·11
XI. 23	14 49	C 8 X 0·5 D	1·18
30	12 30	C 9 X 0·5 D	1·20
1906			
I. 14	11 45	X 1·5 D	1·08
16	13 15	X 2 D	1·03
29	12 35	C 7 X 2 D	1·09
II. 17	11 13	C 12 X 4 D	1·06
25	7 47	C 5·5 X 4 D	0·91 *c*
26	9 30	C 7 X 4 D	0·95 *d*
III. 12	8 25	C 8 X 4 D	0·98
15	8 5	C 9 X 5 D	0·95
25	8 5	C 7·5 X 4·5 D	0·94
XI. 14	15 20	C 5 X 3·5 D	0·92
17	17 55	C 7 X 4 D	0·96
XII. 12	16 59	C 6 X 5 D	0·89
13	11 4	C 2 X 3 D	0·85
1907			
I. 10	12 9	C 2·5 X 6 D	0·68
II. 15	11 55	C 7 X 5 D	0·90
III. 4	9 0	C 6 X 5·5 D	0·87 *c*
10	8 30	C 5 X 5 D	0·85 *e*
XI. 9	16 0	X 5 D	0·73
29	11 57	C 1·5 X 4 D	0·80

Date.	Time. h m		Mag.
1907			
XI. 29	15 5	C 4 X 4 D	0·85
XII. 10	16 40	C 6 X 5 D	1·08
1908			
I. 26	10 10	C 3 X 4 D	0·80
III. 25	8 55	C 5 X 5 D	0·84 *e*
XI. 19	17 52	C 6 X 6 D	0·82
20	11 30	C 2 X 5 D	0·85
,,	15 20	C 1·5 X 7 D	0·60
24	15 35	C 4·5 X 6 D	0·78
XII. 17	14 10	C 5 X 5 D	0·84
1909			
II. 19	8 0	C 3 X 5 D	0·78
20	10 15	C 3·5 X 6 D	0·72
X. 24	17 20	C 6 X 3·5 D	0·95
XI. 9	12 16	C 3 X 2 D	0·98
19	16 10	C 12 X 2·5 D	1·11
XII. 2	15 25	X 2 D	1·03
4	13 20	X=D	1·19
,,	15 30	X 0·5 D	1·21
19	12 30	X 1·5 D	1·04
,,	15 40	C 10 X 2 D	1·17
20	12 25	X 2 D	1·01
1910			
I. 2	15 54	X 2 D	1·36
7	12 55	C 9 X 2 D	1·09
8	15 16	C 15·5 X 2 D	1·29
9	8 55	C 6·5 D 0·5 X 3 E	1·17
18	13 0	X 2 D	1·03
II. 11	11 15	C 10 X 2·5 D	1·10
III. 5	8 55	X 2·5 D	0·99 *f*
X. 13	17 10	X 0·5 D	1·18
XI. 5	16 30	X 1·5 D	1·08
XII. 20	12 55	X=D	1·23
21	9 25	X=D	1·10
24	10 50	X 0·5 D	1·12

a At Ponferrada, Spain, alt. 1780 ft.
b At Vigo, Spain.
c At Carperby, N. Yorkshire, alt. 670 ft.
d At Bainbridge, N. Yorkshire, alt. 800 ft.
e At Torquay.
f At Yarm, Yorkshire.

Date.	Time.		Mag.
1910	h m		
XII. 25	11 40	C 7 X 1 D	1·11
29	12 0	C 7 X 1 D	1·11
1911			
I. 6	12 15	D 0·5 X	1·28
30	10 10	C 7 X=D	1·20
II. 19	8 50	C 10 X=D	1·17 *a*
22	8 20	C 8 X=D	1·20
23	10 30	C 9 X 0·5 D	1·22
26	8 30	D 0·5 X	1·26
III. 4	10 10	X=D	1·23
25	9 55	X 0·5 D	1·34 *b*
XI. 21	10 13	X 1 D	0·87 *c*
XII. 20	10 50	C 5 X 2 D	1·01
,,	12 7	C 6 X 1·5 D	1·02
21	15 25	C 10 X 2 D	1·15
24	10 20	C 8 X 1·5 D	1·07
,,	13 20	C 8 X 1·5 D	1·11
1912			
I. 7	13 15	X 2·5 D	0·98
II. 8	12 40	X 3 D	1·04
III. 6	10 4	C 6 X 5 D	0·88
19	9 40	C 6 X 5 D	0·87
IV. 5	9 15	C 4 X 5 D	0·81 *d*
XI. 10	12 30	X=D	1·10
11	12 20	X=D	1·09
17	15 8	C 7 X 2 D	1·06
19	15 45	C 8·5 X 2 D	1·09
XII. 8	13 40	C 7 X 1 D	1·12
13	10 20	C 5 X=D	1·13
19	16 32	C 12 X 2 D	1·26
29	13 15	X 1·5 D	1·08
1913			
I. 1	14 15	C 10 X 0·5 D	1·25
II. 3	11 25	C 10·5 X 3 D	1·07
6	12 43	C 10 X 2·5 D	1·13
7	11 40	C 10 X 3 D	1·07

Date.	Time.		Mag.
1913	h m		
III. 8	9 5	C 4 X 5 D	0·79 *e*
11	8 20	C 5 X 5 D	0·85 *e*
,,	10 20	C 6 X 5 D	0·88 *e*
X. 26	15 45	C 1·5 X 6 D	0·65
30	17 37	C 7 X 7 D	0·84
XI. 2	16 45	C 4 X 6 D	0·76
24	13 11	C 2·5 X 4·5 D	0·78
XII. 3	14 40	B 1·5 X 6 D	0·73
27	17 7	C 6 X 4·5 D	1·15
31	11 35	C 6 X 4 D	0·92
1914			
I. 1	13 25	C 7 X 4 D	0·94
2	10 2	C 3 X 3·5 D	0·87
,,	11 58	C 4·5 X 3·5 D	0·89
4	14 45	C 7·5 X 4·5 D	1·02
II. 15	9 10	C 4 X 4 D	0·86
XI. 11	11 33	C 0·5 D 1 X	1·20
,,	15 0	X 0·5 D	1·15
23	14 50	X 0·5 D	1·18
24	14 50	X=D	1·23
26	16 4	C 7 D 0·5 X 2·5 G	1·31
XII. 7	16 5	X 1 D	1·13
8	4 40	X 0·5 D	1·18
1915			
I. 5	13 5	X 1·5 D	1·08
7	11 0	C 7·5 X 0·5 D	1·15
,,	12 50	X=D	1·23
16	11 35	X=D	1·23
II. 9	8 25	C 6 X 1 D	1·09
III. 7	8 18	C 8 X 0·5 D	1·13
14	8 5	C 9 D 2 X 3 F	1·42 *e*
15	10 0	C 10 D 1·5 X	1·52 *e*
18	8 10	D 2 X	1·44 *e*
,,	9 0	C 9 D 1 X 1·5 E	1·36 *e*
,,	10 29	X=D	1·42 *e*
19	7 48	X 0·5 D	1·18 *e*

a At West Hartlepool.
b At Great Ayton, N. Yorkshire, alt. 300 ft.
c At Dundee.
d Off Portugal.
e At Torquay.

Date.	Time. h m		Mag.
1915			
III. 19	9 44	X 0·5 D	1·30 *a*
20	9 40	X 1 D	1·25 *a*
28	8 40	D 5 X	1·64
IV 4	8 55	D 0·5 X	1·39
,,	10 9	X 1 D	1·52
5	9 44	C 23·5 X 2 5 D	1·33
7	8 59	D 2 X 8 H	1·49
15	9 12	X 2·5 D	1·28 *b*
,,	9 14	C 16 X 2 D	1·38 *b*
IX. 5	14 0	E 2 X 2 H	1·42
,,	14 30	D 10 X 2 H	1·53
X. 14	16 50	D 6 X 5 H	1·54
XI. 10	11 5	X 2 D	0·89
,,	15 0	X 0·5 D	1·18
14	16 10	X = D	1·23
15	14 25	X 0·5 D	1·18
XII. 2	11 0	C 10 X 1 D	1·06
7	11 24	X = D	1·14
,,	15 45	D 1 X	1·37
8	12 0	C 8 X = D	1·17
12	15 55	D 1·5 X 5·5 H	1·43
26	10 5	X 0·5 D	1·08
31	9 25	D 0·5 X	1·19
1916			
I. 1	14 13	X 0·5 D	1·18
2	7 50	X = D	1·04
,,	14 30	D 0·5 X	1·35
4	8 48	D 0·5 X	1·17
6	14 35	D 0·5 X	1·28
7	13 10	D 0·5 X	1·28

Date.	Time. h m		Mag.
1916			
I. 12	15 3	X 2 D	1·30
22	6 55	D 1 X	1·18
II. 2	8 35	C 10 X 0·5 D	1·14
6	7 50	C 10 X 1·5 D	1·09
7	7 25	X = D	1·15
,,	13 15	X = D	1·23
III. 24	9 8	X 2 D	1·03
31	9 5	X = D	1·23
IV. 4	9 30	D 1 X	1·55
X. 30	11 5	X 1·5 D	0·77
31	10 45	X 1 D	0·68
,,	16 40	X = D	1·23
XI. 1	15 40	X 1 D	1·13
,,	17 30	C 10 X 1 D	1·17
2	11 52	D 1 X	1·17
3	16 45	X 1 D	1·13
20	11 30	C 8 X 1·5 D	1·04
21	15 15	X 1·5 D	1·08
26	10 40	D 0·5 X	1·09
,,	15 30	X 0·5 D	1·18
30	14 15	X 1 D	1·13
XII. 3	15 15	X 1·5 D	1·08
20	14 0	X 1 D	1·13
21	12 10	C 8 X 0·5 D	1·15
25	15 15	C 10 X 2 D	1·16
29	11 30	C 8 2 D	1·06
1917			
II. 12	9 45	C 12 X 1·5 D	1·15
25	8 50	C 12 X 1 D	1·17
III. 21	8 45	C 12 X 2 D	1·15

a At Torquay.

b At Darlington.

Observations of Comets made with the 10-inch Refractor of the University Observatory, Utrecht, Holland. By J. van der Bilt, Ph.D.

1917 *b* (*Schaumasse*).

Date.	G.M.T.	App. R.A.	Parallax Factor.	App. Decl.	Parallax Factor.	No. of Obs. R.A.	No. of Obs. Decl.
1917	h m s	h m s		° ′ ″			
May 26	11 58 41	6 45 31·78	9·516	+53 48 52·6	0·891	8	2
27	10 40 19	7 15 2·75	9·686	+50 45 30·3	0·817	8	2

1916 *b* (*Wolf*).

June 13	11 49 38	22 13 37·33	9·562$_n$	+22 42 37·6	0·752	12	3
14	12 0 44	22 15 44·48	9·551$_n$	+22 51 42·3	0·743	12	3
July 15	13 14 49	23 10 42·20	9·303$_n$	+24 26 13·7	0·645	6	3
19	13 47 28	23 15 57·25	9·124$_n$	+24 7 14·7	0·628	6	3
23	11 10 6	23 20 32·87	9·511$_n$	...	...	4	...
23	11 14 51	...	...	+23 41 14·9	0·710	...	3
25	11 58 48	23 22 44·48	9·425$_n$	+23 24 36·6	0·643	10	5
27	11 24 25	23 24 46·11	9·476$_n$	+23 6 16·6	0·700	20	4
30	11 17 26	23 27 35·11	9·471$_n$	+22 34 39·2	0·704	8	4
Aug. 10	10 19 18	23 35 21·06	9·497^n	+19 58 57·9	0·737	20	5
15	12 20 48	23 37 39·85	9·118$_n$	+18 26 3·6	0·697	12 12	3 3
16	11 53 2	23 38 0·49	9·239$_n$	+18 6 42·2	0·708	12	3
17	12 19 46	23 38 21·05	9·080$_n$	+17 46 7·0	0·702	10	5
18	11 32 49	23 38 39·32	9·269$_n$	+17 26 14·3	0·717	6	10
20	12 59 51	23 39 11·72	9·588$_n$	+16 42 23·7	0·707	5	10
28	12 4 6	23 40 22·33	8·897$_n$	+13 38 31·9	0·740	6	10

Comparison Stars.

Date.	Mean R.A. 1917·0.	Red. to Date.	Mean Decl. 1917·0.	Red. to Date.	Reference.
	h m s	s	° ′ ″	″	
May 26	6 43 13·99	+1·45	+53 51 42·9	+ 3·6	A. G. Cbr. U.S. 2624
27	7 11 56·53	+1·91	+50 39 11·4	+ 6·4	,, 2801
June 13	22 12 46·06	+2·65	+22 28 57·0	+ 4·8	A. G. Berl. B. 8584
14	22 16 47·86	+2·66	+22 38 35·4	+ 5·1	,, 8606

Comparison Stars—continued.

Date.	Mean R.A. 1917·0.	Red. to Date.	Mean Decl. 1917·0.	Red. to Date.	Reference.
	h m s	s	° ′ ″	″	
July 15	23 11 31·28	+3·40	+24 29 17·8	+13·1	A. G. Berl. B. 8902
19	23 13 31·34	+3·52	+24 3 21·7	+14·3	,, 8913
23	23 20 12·73	+3·58	+23 32 19·0	+15·3	, 8961
25	23 21 55·18	+3·62	+23 16 13·1	+15·8	,, 8973
27	23 23 49·46	+3 67	+23 7 42·9	+16·4	,, 8985
30	23 24 57·58	+3·75	+22 35 36·2	+17·3	,, 8993
Aug. 10	23 35 4·22	+3·97	+20 0 12·0	+20·6	Berl. A. 9648
15	23 37 48·72	+4·08	+18 26 58·2	+22·2	B.D.+18°5193 } Referred to comp.
15	23 38 0·20	+4 08	+18 27 16·5	+22·2	B.D.+18°5194 } star of Aug. 16
16	23 38 19·29	+4·10	+18 12 24·8	+22·5	A. G. Berl. A. 9666 (p.m. applied)
17	23 36 4·47	+4·12	+17 45 15·8	+22·8	A. G. Berl. A. 9652
18	23 38 21·06	+4·13	+17 16 22·4	+23·0	,, 9667
20	23 39 15·89	+4·15	+16 45 57·7	+23·5	,, 9672
28	23 40 28·37	+4·28	+13 38 35·6	+25·6	Lpz., 9421

Notes.

Aug. 10. The nucleus is surrounded by a very faint fan-shaped coma.
Aug. 15. The declination of +18°5193 ought to be corrected by +1″·3 in the Bonner Durchmusterung.
Aug. 20. Haze.

Utrecht:
1917 *September* 20.

Occultations of Stars and of the Planet Mars by the Moon, observed with the 10-inch Refractor of the University Observatory, Utrecht, Holland. By J. van der Bilt, Ph.D.

Date.	Star.	Phase.	G.M.T. h m s
1912 Oct. 19	−20° 6194	D	8 10 3·9
Nov. 18	− 7° 5990	D	9 35 12·9
1913 Jan. 27	Spica	D	13 8 24·4
,, 27	,,	R	14 11 5·8
Mar. 15	+28° 853	D	9 42 29·1
,, 15	+28° 854	D	9 44 36·7
June 12	B.A.C. 4200	D	8 58 41·1
,, 12	− 3° 3299	D	9 3 34·7
,, 13	−11° 3502	D	11 1 21·6
July 23	B.A.C. 57	D	11 24 29·6
Sept. 20	17 Tauri	D	12 3 2·0
,, 20	16 ,,	D	12 17 57·8
,, 20	20 ,,	D	12 49 45·3
,, 20	17 ,,	R	13 14 17·5
,, 20	16 ,,	R	13 15 4·9
,, 20	+23° 512	R	13 49 54·4
,, 20	20 Tauri	R	13 53 4·0
,, 24	*c* Geminorum	D	14 52 49·8
Oct. 17	66 Arietis	R	11 3 3·0
Dec. 19	τ Leonis	D	12 58 2·9
,, 19	83 ,,	D	13 23 57·2
,, 19	τ ,,	R	14 1 29·8
,, 19	83 ,,	R	14 3 30·4
1914 Feb. 10	Regulus	D	17 52 32·5
Apr. 3	+26° 1463	D	9 52 2·9
May 2	+20° 2219	D	11 9 55·8
,, 30	Mars	D_1	5 8 5·8
,, 30	,,	D_2	10·6
,, 30	,,	R_1	6 21 11·6
,, 30	,,	R_2	19·8
1915 Mar. 25	W.Z.C. 550	D	11 54 41·6
Apr. 22	,, 582	D	10 46 38·7
Oct. 2	Mars	D_1	12 13 7·1
,, 2	,,	D_2	20·1
,, 2	,,	D_3	28·1
,, 2	,,	R_1	12 52 13·6
,, 2	,,	R_2	25·1
,, 2	,,	R_3	41·6
Dec. 18	16 Tauri	D	11 58 12·1
,, 18	19 ,,	D	12 15 21·7
,, 18	20 ,,	D	12 34 4·6
,, 18	21 ,,	D	12 45 59·5
,, 18	22 ,,	D	12 47 40·4
,, 18	19 ,,	R	13 26 58·9
,, 18	20 ,,	R	13 46 6·2

The occultation of *c* Geminorum was observed by Professor Nijland. In the observations of Mars the indices 1 and 2 on 1914 May 30 refer to the limbs; on 1915 October 2 the indices 1 and 3 refer to the limbs, the index 2 to the centre of the disk. The time given under R_3 is uncertain, since the dark limb of the moon became invisible at the place of contact.

Utrecht:
1917 *September* 20.

Observations of Comets made with the 10-inch Refractor of the University Observatory, Utrecht, Holland. By G. van Biesbroeck.

(*Communicated by J. van der Bilt.*)

1916 *b* (*Wolf*).

Date.	G.M.T.	App. R.A.	Parallax Factor.	App. Decl.	Parallax Factor.	No. of Obs. R.A.	No. of Obs. Decl.
1917	h m s	h m s		° ′ ″			
Jan. 20	17 49 46	16 41 57·70	9·348$_n$	− 5 42 59·7	0·863	20	5
Feb. 2	17 43 38	17 8 43·21	9·299$_n$	− 4 57 29·4	0·862	6	6
Mar. 21	16 22 58	18 54 22·60	9·301$_n$	+ 1 56 59·0	0·829	5	5
23	16 5 51	...	...	+ 2 24 5·3	0·828	...	3
23	16 16 7	18 59 6·42	9·312$_n$	...	...	3	...
24	16 0 11	19 1 27·93	9·349$_n$	+ 2 37 11·9	0·827	6	6
Apr. 1	15 11 19	19 20 35·11	9·425$_n$	+ 4 30 55·9	0·819	6	6
May 1	14 31 52	20 33 44·43	9·425$_n$	+12 38 1·5	0·772	6	6
2	13 51 33	20 36 7·13	9·486$_n$	+12 54 18·1	0·782	6	6
3	14 0 55	20 38 34·67	9·471$_n$	+13 11 2·8	0·777	6	6
13	13 7 42	21 2 43·88	9·522$_n$	+15 55 4·4	0·775	6	6
20	13 46 47	21 19 33·52	9·463$_n$	+17 43 51·2	0·742	6	6

1917 *a* (*Mellish*).

Mar. 23	7 39 46	2 8 15·28	9·574	+15 22 32·6	0·822	6	8
24	7 19 36	2 7 34·18	9·573	+15 43 14·4	0·814	14	7
27	6 59 0	2 4 35·53	9·574	+16 45 15·9	0·809	6	6
30	6 54 44	...	...	+17 41 54·1	0·812	...	3

1917 *b* (*Schaumasse*).

May 14	14 19 7	23 59 43·34	9·635$_n$	+33 8 2·5	0·754	6	6
17	13 56 59	0 38 6·78	9·684$_n$	+42 31 34·5	0·758	6	6
19	12 49 7	1 22 45·11	9·655^n	+49 51 20·4	0·839	6	6
20	13 20 20	1 56 41·72	9·692$_n$	+53 30 36·4	0·822	6	6
22	10 34 0	3 23 22·41	8·941	+58 33 27·7	0·913	6	6

Comparison Stars.

Comet 1916 *b*.

Date.	Mean R.A.	Red. to Date.	Mean Decl.	Red. to Date.	Reference.
	h m s	s	° ′ ″	″	
Jan. 20	16 41 16·91	+0·37	− 5 48 0·0	−8·0	A. G. Strsb. 5739
Feb. 2	17 8 54·20	+0·62	− 5 3 9·1	−9·0	,, ,, 5840
Mar. 21	18 54 34·22	+1·41	+ 2 0 19·6	−9·3	,, Alb. 6454
23	18 59 24·86	+1·41	+ 2 25 23·5	−9·1	,, ,, 6496
24	19 1 5·19	+1·43	+ 2 37 19·2	−9·1	,, ,, 6519
Apr. 1	19 20 57·81	+1·53	+ 4 25 49·1	−8·3	,, ,, 6667
May 1	20 34 12·53	+1·91	+12 38 23·9	−4·1	,, Lpzg. 8047
2	20 35 55·34	+1·93	+12 52 14·0	−4·0	,, ,, 8066
3	20 38 40·15	+1·94	+13 17 13·8	−3·8	,, ,, 8098
13	21 2 31·80	+2·08	+15 52 24·6	−1·9	,, Berl. A. 8583
20	21 19 1·87	+2·20	+17 42 24·5	−0·5	,, ,, 8722

Comet 1917 *a*.

Mar. 23	2 8 4·81	+0·62	+15 24 37·0	+5·6	Bord. ph. Cl. 350, No. 49
24	2 8 32·98	+0·62	+15 41 50·6	+5·6	,, No. 52
27	2 4 48·83	+0·59	+16 50 16·6	+5·7	A. G. Berl. A. 606
30	1 59 9·32	...	+17 51 18·5	+5·4	,, ,, 587

Comet 1917 *b*.

May 14	23 59 40·26	+1·11	+33 1 27·5	+0·5	A. G. Lei. 10218
17	0 38 9·62	+0·93	+42 27 10·3	+0·4	,, Bonn 551
19	1 22 12·00	+0·78	+49 57 11·5	+1·1	Bo. VI. (B.D. +49° 383)
20	1 56 29·93	+0·70	+53 36 59·0	+2·0	A. G. Cbr. U.S. 938
2	3 23 16·48	+0·71	+58 35 31·2	+4·6	Boss Prel. Gen. Cat. 786

Notes.

1916 *b*. During the period covered by the observations the brightness increased very slowly from about 11^m to about 10^m. A short faint tail was seen on the preceding side of the badly defined nucleus.

1917 *a*. Mar. 23: Brightness (tail included) exceeded 19 Arietis ($5^m·8$). Mar. 24: Bright straight tail extending 15′ in position-angle 55°. Total brightness about $5^m·1$ ($0^m·7 >$ 19 Arietis). Apr. 4: Tail in position-angle 50°. Total brightness about $3^m·6$ ($0^m·5 >\gamma$ Arietis).

1917 *b*. Brightness as observed in 3-in. finder with reference to Potsdam stars: May 19, $6^m·7$; May 20, $7^m·4$; May 22, $7^m·4$.